École Supérieure en Sciences

Appliquées de Tlemcen (ESSAT)

Courses

Modeling

MOKHTARI Mohammed Rida

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Introduction

The optimization and control of energy systems represent a strategic challenge of major importance, both from an industrial and a scientific standpoint. With the continuous rise in energy demand, the pressing need for higher efficiency, and the global imperative to minimize environmental impact, it has become essential to develop advanced tools capable of effectively modeling, analyzing, and monitoring such systems. A robust model should not merely reproduce observed behavior; it must also accurately capture the system's internal structure, thereby providing deep insight into its dynamic properties and internal interactions.

Within this framework, structural analysis stands out as a rigorous and autonomous methodology, independent of exact numerical parameters. It enables the study of a system based solely on its topological configuration and interconnections, identifying measurable variables, computable variables, and those that remain inaccessible. This approach is particularly valuable for assessing the controllability and observability potential of a system before performing simulations or experimental tests.

The first chapter of this work introduces the classical structural analysis approach. This method relies on a conventional mathematical representation—typically through differential or algebraic equations—that characterizes the system's dynamic behavior. Although it provides a solid theoretical basis for understanding model logic, it may become less intuitive when applied to complex, multi-energy systems, where different energy domains interact simultaneously.

To overcome these limitations, the Bond Graph formalism is introduced in the second chapter. This unified modeling language is founded on the principle of power exchange. Its major strength lies in its ability to describe multi-domain systems—electrical, mechanical, thermal, and hydraulic—within a single coherent framework. By explicitly defining energy flows and interconnection ports, this approach promotes a global and unified understanding of system operation and naturally extends toward structural analysis.

The third chapter combines the Bond Graph formalism with structural analysis, forming a key integrative stage that merges the descriptive power of the Bond Graph with the analytical depth of the structural approach. This synergy enables the extraction of the model's internal structure while preserving a comprehensive

energetic perspective—an essential aspect for complex energy systems characterized by strong inter-subsystem interactions.

Finally, the fourth chapter explores the concept of the degree of observability, a fundamental notion in control and supervision theory. Observability defines the ability to reconstruct a system's internal state from its measurable outputs. Quantifying this property makes it possible to evaluate sensor performance, detect potential observability gaps, and design more efficient diagnostic and control strategies.

CHAPTER 1

Modeling Theory

1.1 Introduction

"Industrial companies need to model and simulate their systems from the design stage. Modeling represents the overall process that allows mathematics to be introduced into a science based on experience or observation. The systems under consideration are dynamic, similar to natural phenomena, meaning they evolve over time—their behaviors change according to internal variables and external inputs. These systems are generally multi-physical in nature (mechanical, electrical, hydraulic, thermal, etc.). Modeling plays a crucial role in system analysis, as it is essential to have a thorough understanding of the physical system's behavior and its environment. Numerical simulation, on the other hand, is the process of computing the solutions of these models using a computer, in order to reproduce physical reality as accurately as possible.

1.1.1 What is a System

A system can be defined as a set of elements that interact with each other according to certain principles or rules. It is considered an entity that can be separated from the rest of its environment by physical and/or conceptual boundaries. Systems can receive information and energy from the external world and can also transmit information and/or energy to it.

1.1.2 What is Modeling

Modeling a system consists of developing a mathematical representation that allows describing and predicting its dynamic and steady-state behavior when it is subjected to external influences (such as control inputs or disturbances).

The modeling process of a system can be divided into three main parts.

- Define the physical phenomena of the system to be controlled.

- Establish an inventory of the measurable variables and identify the origin of the main phenomena (hydraulic, electrical, mechanical, etc.).

- The formulation of the system equations can be presented in different forms.

1.1.3 What is Simulation

Simulation computes the theoretical evolution (or response) of the physical quantities of a system under the effect of external inputs or disturbances. Its objective is to predict the behavior of a system using simulation models (also referred to as virtual prototypes).

1.1.4 What is a Model

A model is a description of a system and its environment based on mathematical objects defined within a given theoretical framework. The components of the model are built upon assumptions that are associated with a specific study objective. These modeling assumptions define the domain of validity of the model.

By nature, a model can never be completely true (inherent incompleteness). However, it can be useful, and the system becomes easier to optimize and control. The development of a model is ensured through:

- The study of the physics of the system (kinematic and dynamic equations, etc.)
- The formulation within an appropriate formalism:
  - Input/output differential equation → Laplace transform → frequency-domain methods.
  - State evolution equation → state-space representation → time-domain methods.

The graphical representation of a model is shown in Figure 1.1.

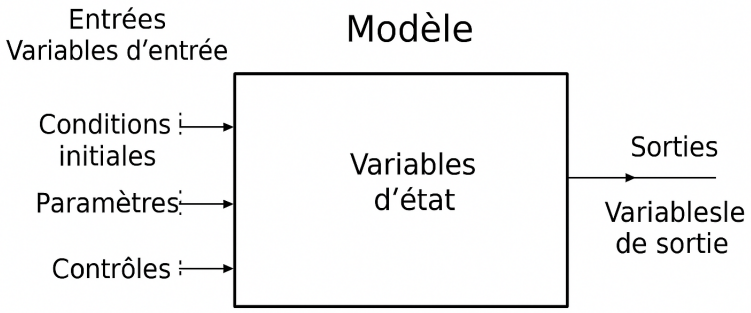


Figure 1.1: Typical diagram of a model

`State variables`: quantities that describe the state of the studied system at time t.

`Structural variables`: quantities (continuous or discrete) that define the structure of the model.

`Parameters`: quantities (constant or time-dependent) that express the relationships between state variables and with the environment of the studied system.

We can distinguish:

- `Known parameters`: obtained, for example, from physical laws.
- `Unknown parameters`: to be determined through direct experimentation or by comparing the model with experimental data.
- `Controls (or control variables)`: quantities that can be manipulated and act upon the system.

A model of the system and its solution become a powerful tool in the hands of system designers. They can be used to answer the following questions:

- **Analysis**: For a given input and a known model (with known state variables), what would be the output?
- **Identification**: Given the history of inputs and outputs, what are the variables or parameters of the model?
- **Synthesis**: For a given input and a desired output, can a system be designed so that it behaves in the desired manner?

For the **design of control laws**, models always involve certain simplifications resulting from the assumptions made by control engineers. The assumptions commonly used in system modeling are:

- `Neglecting small effects`: effects that have a relatively minor influence compared to the dominant ones are ignored.
- `Independent environment`: the environment is assumed not to be affected by what happens within the system.
- `Localized characteristics`: the physical properties of the system components are assumed to be lumped, even though they may actually be spatially distributed.
- `Linear relationships`: the constitutive relations are assumed to be linear over the entire operating range of the system, even though in reality they are not exactly linear.
- `Constant parameters`: the parameters defining the properties of the components are assumed to remain constant.
- `Neglecting uncertainty and noise`: any uncertainty or noise in the data is disregarded.

Following these assumptions, the equations governing the system model turn out to be a set of linear ordinary differential equations with constant parameters. The solutions of these ordinary differential equations are relatively easier to obtain and describe the dynamic behavior of the system.

If these simplifying assumptions were not made, the equations would become a set of nonlinear partial differential equations with parameters varying over time and space. Such a set of equations might yield a more accurate mathematical model of the system, but it would not be very practical, since these types of equations are much more difficult to solve. Without efficient and effective solution techniques, the model would fail to provide useful results for engineers. Thus, the benefits of simplification far outweigh the information lost due to these assumptions.

1.2 Types of Models

Two modeling approaches are used in the development of multi-physical system models.

1.2.1 Knowledge-Based Model

A knowledge-based model is generally developed either from the fundamental laws of physics (mechanics, electromagnetism, thermodynamics, etc.) or from empirical relationships. These models are expressed in the form of differential equations, whose parameters represent characteristic quantities of the phenomenon under study.

1.2.2 Behavioral Model

A behavioral model is represented as a computer program that can be executed on dedicated software. It is expressed in the form of a mathematical relationship that provides the time response as a function—among other things—of various factors, different from the physical parameters, which have no direct physical link with the phenomenon being studied.

1.3 The Different Forms of Modeling

1.3.1 Input/Output Model: The Differential Equation

To simplify a linear model and make it more compact, a common approach is to combine all the equations into a single one. This involves eliminating all the internal variables of the system that are neither input nor output. The result is a single differential equation involving only the input u, the output y, and their successive derivatives. Such an equation takes the following form:

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \dot{y}(t) + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \dot{u}(t) + b_0 u(t) \tag{1.1}$$

This is an input/output behavioral model, which expresses the evolution of the output as a function of the input.

Physically, it is necessary that m be strictly less than or equal to n; otherwise, it would mean that the output of the system at a given instant depends on the value of the input at a later instant. Such a system is said to be **causal**.

Example: Let us take as an example an RLC circuit.

The equations derived from the laws of electricity that govern the behavior of the RLC circuit are as follows:

$$\begin{cases} u(t) &= Ri(t) + L\frac{di(t)}{dt} + y(t) \\ y(t) &= \frac{1}{C}\int_0^t i(\tau)d\tau \;\Leftrightarrow\; \frac{dy(t)}{dt} = \frac{1}{C}i(t) \end{cases} \tag{1.2}$$

By combining the two equations given in (1.2), we obtain, after eliminating $i(t)$:

$$u(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) \tag{1.3}$$

This single differential equation thus constitutes a model representing the relationship between the input $u(t)$ and the output $y(t)$.

1.3.2 Transfer Function

The transfer function is an input/output behavioral model obtained from the linear differential equation with constant coefficients. This model contains no information about the physical structure of the system. Rather than seeking to express $y(t)$ as a function of $u(t)$, the objective is to obtain $Y(s) = \mathcal{L}\{y(t)\}$ as a function of $U(s) = \mathcal{L}\{u(t)\}$, where $\mathcal{L}(\cdot)$ denotes the Laplace transform.

By applying the Laplace transform to the differential equation (1.1) (assuming zero initial conditions), we obtain the following expression:

$$\frac{Y(s)}{U(s)} = H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{1.4}$$

- $H(s)$, called the *transfer function*, depends on the Laplace variable s, with numerator $N(s)$ and denominator $D(s)$.
- The roots of $N(s)$ are called the *zeros* of the system, while the roots of $D(s)$ are called the *poles* of $H(s)$.
- The system is said to be of order n, and according to the causality rule, we generally have $m \leq n$.

Example: In the case of the RLC circuit, applying the Laplace transform yields:

$$U(s) = LC\left(s^2Y(s) - sy(0) - \dot{y}(0)\right) + RC\left(sY(s) - y(0)\right) + Y(s) \tag{1.5}$$

Hence,

$$\begin{array}{rcccl} Y(s) &=& \frac{1}{LCs^2+RCs+1}U(s) &+& \frac{LCsy(0)+(LC+RC)\dot{y}(0)}{LCs^2+RCs+1} \\ Y(s) &=& \frac{N(s)}{D(s)}U(s) &+& \frac{I(s)}{D(s)} \end{array} \tag{1.6}$$

The function $H(s) = \frac{N(s)}{D(s)}$ allows us to determine the term of $Y(s)$ that depends solely on $U(s)$, independently of the initial conditions, which appear in the numerator $I(s)$.

1.3.3 State Representation

The concept of the *state* is a mathematical tool designed to facilitate the study of a system's behavior. The state representation is often described as a modeling approach in the *state space*.

[`State Variable`] These are the variables or quantities that define the state of the system.

[`State Vector`] A state vector is the minimal set of state variables—that is, temporal quantities—necessary and sufficient to determine the future evolution of a system when the equations governing the system's operation and its inputs are known.

A state vector and its derivative are denoted as:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} \tag{1.7}$$

The number n of components corresponds to the order of the system.

[`State Equation`] In the general case, the state-space representation of most physical systems is given in the following nonlinear form:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = f(x(t), u(t), t) = \begin{bmatrix} f_1(x(t), u(t), t) \\ \vdots \\ f_n(x(t), u(t), t) \end{bmatrix} \\ y(t) = g(x(t), u(t), t) \end{cases} \tag{1.8}$$

where f and g are functions that can take almost any form.

From Nonlinearity to Linearity

The two functions f and g are generally nonlinear; therefore, a linear approximation is often used. To achieve this, it is commonly assumed that the system's state and input evolve in the vicinity of an operating point (equilibrium point), and the nonlinear model can be expressed in the linear form:

$$\begin{cases} \dot{x}(t) &= Ax(t) &+ Bu(t) \\ y(t) &= Cx(t) &+ Du(t) \\ x(t_0) &= x_0 \end{cases} \tag{1.9}$$

The relations given in (1.9) represent a linear time-invariant (LTI) state-space model of a system.

- The matrix $A \in \mathbb{R}^{n \times n}$ is called the *state* or *system* matrix.
- The matrix $B \in \mathbb{R}^{n \times m}$ is called the *input* or *control* matrix.
- The matrix $C \in \mathbb{R}^{p \times n}$ is called the *output* or *observation* matrix.
- The matrix $D \in \mathbb{R}^{p \times m}$ is called the *direct transmission* matrix.

The state-space representation can be associated with the block diagram shown in Figure 1.2.

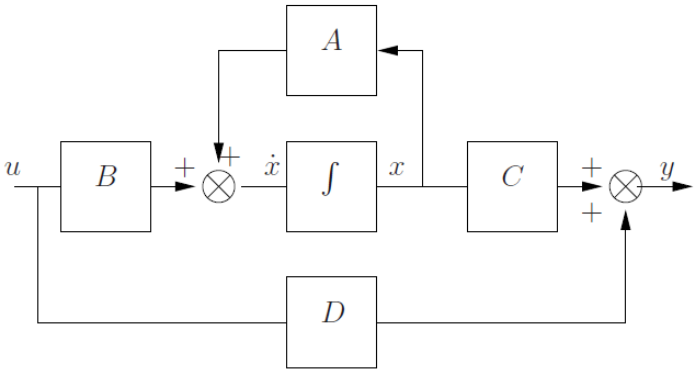


Figure 1.2: The state-space representation

Remark

In the special case where $p = m = 1$, i.e., there is a single input and a single output, the system is called *single-input single-output (SISO)* or *monovariable*. Otherwise, it is called *multi-input multi-output (MIMO)*.

Example: (**Mass–Spring–Damper System**) Consider the motion of a car illustrated in Figure **??**, under the following assumptions:

- The car moves forward.
- The accelerator/brake force is given by $F = U\vec{z}$.
- Only aerodynamic drag is considered: $F_a = -fv^2\vec{z}$.

The kinematic equation of translation is:

$$\frac{dz}{dt} = v \tag{1.10}$$

Applying Newton's second law of motion gives:

$$\frac{dv}{dt} = \frac{1}{m}U - \frac{f}{m}v^2 \tag{1.11}$$

By choosing the state vector with $x_1 = z$ and $x_2 = v$, the control input $u = U$, and the measurement vector $y^T = [z_m \; v_m]$ (assuming that both position and velocity are measured), the nonlinear state equation becomes:

$$\begin{cases} \dot{x}_1 = x_2 & \text{(i.e., } f_1(x_1, x_2)) \\ \dot{x}_2 = \frac{1}{m}u - \frac{f}{m}x_2^2 & \text{(i.e., } f_2(x_1, x_2)) \end{cases} \tag{1.12}$$

The equilibrium point is chosen arbitrarily as $\{v_e, u_e\}$. The linearized model around this equilibrium point is:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{2fv_e}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u \tag{1.13}$$

The observation equation (assuming the entire state vector is measured) is:

$$\begin{pmatrix} x_{1m} \\ x_{2m} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.14}$$

How to Obtain a State-Space Model

A linear state-space representation can be derived either from a set of component equations or directly from a single differential equation.

Series RLC Circuit: State-Space and Transfer Function

Consider a series RLC circuit. By applying Kirchhoff's voltage law around the loop, we have

$$v_s(t) = v_R(t) + v_L(t) + v_C(t),$$

with

$$v_R(t) = Ri(t), \quad v_L(t) = L\frac{di(t)}{dt}, \quad v_C(t) = \frac{1}{C}\int i(t)\,dt.$$

Differentiating to eliminate the integral term, the system satisfies the second-order differential equation:

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv_s(t)}{dt}. \tag{1.15}$$

Transfer Function. Taking the Laplace transform of (1.15) (zero initial conditions) gives:

$$\frac{I(s)}{V_s(s)} = \frac{s}{Ls^2 + Rs + \frac{1}{C}}, \quad \frac{V_C(s)}{V_s(s)} = \frac{1}{Ls^2 + Rs + \frac{1}{C}}. \tag{1.16}$$

State-Space from a Single Differential Equation. For a general n-th order differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_0 u,$$

define the state variables

$$x_1 = y,\ x_2 = \dot{y},\ \ldots,\ x_n = y^{(n-1)}.$$

Then the state-space matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \quad D = 0.$$

Applying this to the RLC equation

$$u(t) = LC\ddot{y}(t) + RC\dot{y}(t) + y(t)$$

gives

$$\ddot{y} + \frac{R}{L}\dot{y} + \frac{1}{LC}y = \frac{1}{LC}u,$$

with state-space matrices

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{1}{LC} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0.$$

Mechanical System Example: Mass–Spring–Damper

Consider a mass m connected to a spring k and damper b, subject to external force $F(t)$:

$$m\ddot{y} + b\dot{y} + ky = F(t).$$

Define the state vector $x = [x_1, x_2]^T$ with $x_1 = y$ and $x_2 = \dot{y}$. The state-space equations are

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x. \tag{1.17}$$

The transfer function from force $F(t)$ to position $y(t)$ is

$$H(s) = \frac{Y(s)}{F(s)} = \frac{1}{ms^2 + bs + k}.$$

From State-Space to Transfer Function

For any linear system $\dot{x} = Ax + Bu,\ y = Cx + Du$, the transfer function is obtained as

$$H(s) = C(sI - A)^{-1}B + D.$$

1.4 Controllability

1.4.1 Definition and Principle

Controllability is a fundamental property of dynamical systems, particularly within the state-space framework. It characterizes the ability, through suitable control inputs, to steer the system state from any initial condition to any desired final condition within a finite time interval.

Consider a continuous-time linear system of the form:

$$\dot{x} = Ax + Bu \tag{1.18}$$

The system is said to be **controllable** if, for any time interval $[t_i, t_f]$ and for any initial and final states x_i and x_f, there exists an admissible control input $u(t)$ such that:

$$x(t_i) = x_i, \qquad x(t_f) = x_f.$$

1.4.2 Kalman Controllability Criterion

A necessary and sufficient condition for the controllability of the pair (A, B) is given by the **Kalman rank criterion**:

$$\text{rank}(\Gamma) = n,$$

where n denotes the dimension of the state vector and Γ is the **controllability matrix**:

$$\Gamma = [\, B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B \,] \tag{1.19}$$

If $\text{rank}(\Gamma) = n$, then every state variable can be reached from the control input through an appropriate control signal.

1.4.3 Geometric Interpretation

The **controllable subspace** $\mathcal{X}_c$ is defined as the set of all states that can be reached by the control inputs. It is the subspace generated by the successive images of the matrix B under the action of A:

$$\mathcal{X}_c = \text{span}\{B,\, AB,\, A^2B,\, \ldots,\, A^{n-1}B\}.$$

The system is said to be **completely controllable** if and only if

$$\mathcal{X}_c = \mathbb{R}^n,$$

that is, when the controllable subspace spans the entire state space.

1.4.4 Uncontrollable Modes

If a system is not controllable, some **modes** (eigenvalues of A) cannot be influenced by the input $u(t)$. These are referred to as **uncontrollable modes** and correspond to dynamics that are decoupled from the control input. In practice, such modes limit achievable performance and may lead to instability if not properly addressed.

1.4.5 Popov–Belevitch–Hautus (PBH) Test

An alternative controllability test is the **PBH criterion**, which states that the system (A, B) is controllable if and only if, for all $s \in \mathbb{C}$,

$$\text{rank}\begin{bmatrix} sI - A & B \end{bmatrix} = n.$$

This criterion provides a compact and efficient way to verify controllability, particularly for numerical or symbolic analysis.

1.4.6 Importance of Controllability

Controllability is essential in control system design. A system that is not controllable cannot be arbitrarily driven by its inputs, which directly restricts achievable performance, stability margins, and robustness. Consequently, controllability analysis is a mandatory preliminary step before the synthesis of controllers or observers.

Example 3: Mechanical System — Mass–Spring–Damper (Controllability)
Consider a single mass m attached to a spring k and a damper b. A force input $u(t)$ acts directly on the mass. This second-order system is suitable for illustrating controllability.

By Newton's law:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = u(t).$$

Define the state variables $x_1 = x$ and $x_2 = \dot{x}$. Then,

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\dfrac{k}{m}x_1 - \dfrac{b}{m}x_2 + \dfrac{1}{m}u. \end{cases}$$

In matrix form:

$$\dot{x} = Ax + Bu, \quad A = \begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \dfrac{1}{m} \end{pmatrix}.$$

For $m = 1\,\text{kg}$, $k = 2\,\text{N/m}$, and $b = 0.5\,\text{Ns/m}$:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The controllability matrix is

$$\Gamma = [\, B \;\; AB \,] = \begin{pmatrix} 0 & 1 \\ 1 & -0.5 \end{pmatrix}.$$

Its determinant is $\det(\Gamma) = -1 \neq 0$, hence $\operatorname{rank}(\Gamma) = 2$. Since

1.5 Conclusion

This chapter presented the fundamentals of dynamic system modeling and structural analysis. Key modeling approaches—input–output equations, transfer functions, and state-space representations—were introduced, with emphasis on the state-space framework for modern control. Mechanical and electrical examples illustrated model derivation from physical laws. Controllability and observability were discussed as essential properties for controller and observer design. The next chapter extends these concepts using Bond Graph formalism, providing a unified energy-based modeling approach for multi-domain systems.

CHAPTER 2

Bond Graph

2.1 Introduction

The use of a model is often essential in certain branches of engineering. Moreover, when dealing with complex systems, a parametric model is necessary if the application requires physical knowledge. In such cases, an important step in the modeling phase is the selection of an appropriate model.

The **block diagram** is a tool used to describe and understand the dynamic behavior of a system defined by differential equations, which are transformed into algebraic equations using the Laplace transform. This makes it easy to determine the transfer function of the system, its zeros, poles, and its time and frequency behavior in the case of linearized systems. It is also worth noting the usefulness of a block diagram for studying system control. The connections between the blocks in a block diagram "carry" only a single piece of information and, therefore, can only represent energy transfers within a system in an incomplete way.

Other tools, such as bond graphs, can be used to obtain a more accurate modeling of the structure of the system and the energy exchanges between components. Indeed, in **bond graph** modeling, the links between elements "carry" two types of information whose product represents an energy flow, i.e., power. Based on analogies between physical domains, bond graph modeling naturally allows for multiphysics description using "invariant" elements that are common in different domains (active elements, passive elements, junctions). Information links are also involved in the modeling of measurement and control of systems.

2.2 What is a Bond Graph?

The **Bond Graph (BG)** tool is a unified graphical language used across all fields of engineering science, recognized as a structured approach for the modeling and simulation of multidisciplinary systems.

Modeling a technical system using bond graphs does not require explicitly writing the general conservation laws. Instead, it is essentially based on the characterization of energy exchange phenomena within the system.

The Bond Graph approach allows for modeling that is simultaneously:

- Functional: through its representation using word bond graphs.
- Structural: by visualizing causality properties.

- Behavioral: by deriving mathematical models (such as transfer functions in the linear case, linear or nonlinear state equations, or differential systems) directly from the bond graph model.

It serves as an intermediate framework between the physical system and its associated mathematical model.

2.3 Basic Concepts

The **Bond Graph approach** is a graphical modeling tool based on the concept of analogy. It assumes the presence of localized parameters within the system. This method also makes the cause-and-effect relationships (**causality**) explicitly visible and enables the systematic construction of mathematical models.

The variables in the basic bond graph form the 'tetrahedron of state', as represented in Figure 2.1, and consist of:

- effort *e(t)*
- flow *f(t)*
- time integral of effort $p(t) = \int_0^t e(t)dt$
- time integral of flow $q(t) = \int_0^t f(t)dt$

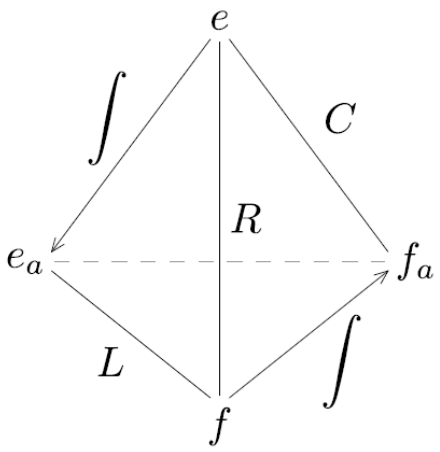


Figure 2.1: Tetrahedron of state'by Henry Paynter.

2.3.1 Multiport Elements

In a bond graph, the multiport elements are the basic building blocks (or nodes) used to represent the physical components of a system. They are represented in the diagram by alphanumeric symbols, each one having a specific meaning.

Table 2.1: List of Common Multiport Elements in Bond Graphs

Symbol	Name	Type	Physical Meaning / Role
R	Resistor	One-port	Represents **energy dissipation** such as friction, electrical resistance, or mechanical damping.
I	Inertia	One-port	Represents **energy storage in kinetic form**, e.g., a mass in mechanics or an inductor in electrical systems.
C	Capacitor (Compliance)	One-port	Represents **energy storage in potential form**, e.g., a spring in mechanics or a capacitor in electrical systems.
TF	Transformer	Two-port	Represents **power transfer with a fixed ratio** between efforts and flows (e.g., gear, ideal transformer).
GY	Gyrator	Two-port	Represents **power conversion** between effort and flow across domains (e.g., electric motor, generator).
0	0-junction	Multiport (n ports)	Represents a **common effort junction** where all connected elements share the same effort and the sum of flows equals zero.
1	1-junction	Multiport (n ports)	Represents a **common flow junction** where all connected elements share the same flow and the sum of efforts equals zero.

2.3.2 Bonds

In a bond graph, the **bonds (power links)** represent the **power** or the **energy flow** exchanged between the two connected multiports. This power is expressed as the product of two complementary variables, called the **effort** and the **flow** variables.

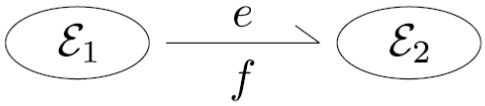
Instantaneous power entering or leaving a port as a function of time is given by the product of effort $e(t)$ and flow $f(t)$:

$$P_u(t) = e(t) \times f(t) \tag{2.1}$$

The direction of positive power is represented by the orientation of the half-arrow.

A bond graph representation provides four key pieces of information:

1. The physical connection between two systems through the bond link.



2. The causality, represented by the causal stroke.
3. The type of energy involved, identified by the nature of the power variables.
4. The direction of transmitted power, indicated by the direction of the half-arrow.

2.3.3 Energy and Power Variables

The product of effort and flow represents power; such variables are called *power variables*.
In addition to power variables (effort flow), two other types of variables are important for dynamical description of systems. These variables are called *energy variables* and are defined as follows:

- $p(t) = \int^t e(t)\,dt \Rightarrow$ generalized momentum or impulse (mechanical impulse, magnetic flux, integral of power),
- $q(t) = \int^t f(t)\,dt \Rightarrow$ generalized displacement.

These two relationships can also be expressed differently using their differential form:

$$\frac{dp(t)}{dt} = e(t) \quad \Rightarrow \quad dp = e(t)\,dt \tag{2.2}$$

$$\frac{dq(t)}{dt} = f(t) \quad \Rightarrow \quad dq = f(t)\,dt \tag{2.3}$$

The energy $E(t)$ is obtained by integrating the power:

$$E(t) = \int^t P_a(t)\,dt = \int^t e(t)f(t)\,dt \tag{2.4}$$

Using relations (4) and (5), the energy can also be expressed as a function of one of the energy variables:

$$E(t) = \int^t e(t)\,dq(t) = \int^t f(t)\,dp(t) \tag{2.5}$$

Consequently, the effort variable $e(t)$ and the flow variable $f(t)$ can be expressed as functions of displacement q and momentum p, respectively: $e = e(q)$ and $f = f(p)$.

Thus, the energy E can be expressed not only as a function of time, but also as a function of one of the energy variables p or q:

$$E_p(q) = \int^q e(q)\,dq \tag{2.6}$$

$$E_c(p) = \int^{p} f(p)\, dp \tag{2.7}$$

Example : The potential energy E_p stored in a spring with stiffness k, assuming that the force (effort) is proportional to the displacement x (energy variable q), is given by:

$$F = kx$$
$$E_1(p) = \int_{q_0}^{q_1} kx\, dx = \frac{1}{2}k(x_1^2 - x_0^2) \tag{2.8}$$

The same approach can be applied to other physical phenomena.

Table 2.2: Effort, flow, and energy variables (Part 1: Electricity, Linear and Rotational Mechanics)

Variable Type	**Electricity**	**Linear Mechanics**	**Rotational Mechanics**
Effort	Voltage $U(V)$	Force $F(N)$	Torque $\Gamma(N \cdot m)$
Flow	Current $I(A)$	Linear velocity $\dot{x}(m/s)$	Angular velocity ω(rad/s)
Momentum	Magnetic flux Φ(Wb)	Mechanical impulse $J(N \cdot s)$	Angular momentum $\Omega(N \cdot m \cdot s)$
Displacement	Charge $q(C)$	Elongation $x(m)$	Angle Θ(rad)

Table 2.3: Effort, flow, and energy variables (Part 2: Hydraulics, Thermal, and Chemical Domains)

Variable Type	**Hydraulics**	**Thermal**	**Chemical**
Effort	Pressure P(Pa)	Temperature $T(K)$	Chemical potential $\mu(J/\text{mol})$
Flow	Volumetric flow $Q(m^3/s)$	Entropy flow $\dot{s}(J/(K \cdot s))$	Molar flow $\dot{n}$(mol/s)
Momentum	Pressure impulse $P_p(N \cdot s/m^2)$	–	–
Displacement	Volume $V(m^3)$	Entropy $s(J/K)$	Number of moles n(mol)

2.4 Bond Graph Language Elements

In this section, we define the set of multiports required for the generic modeling of a physical system using the effort (e) and flow (f) variables.

These basic variables are classified into:

- Three passive elements (R, C, and I).
- Two active elements (Se and Sf).
- Four junctions (1, 0, TF, and GY).

2.4.1 Passive Elements

The elements are called passive because they transform the supplied power into:

- Dissipated power (element R).
- Power stored as potential energy (element C).
- Power stored as kinetic energy (element I).

Resistive Element (R)

The one-port element R represents any phenomenon that **dissipates energy**. In physical systems, this means that the supplied power is not stored but is instead converted—most commonly into **heat**. Figure 2.2 shows some examples of these types of devices in different domains. In bond graph modeling, the resistive element

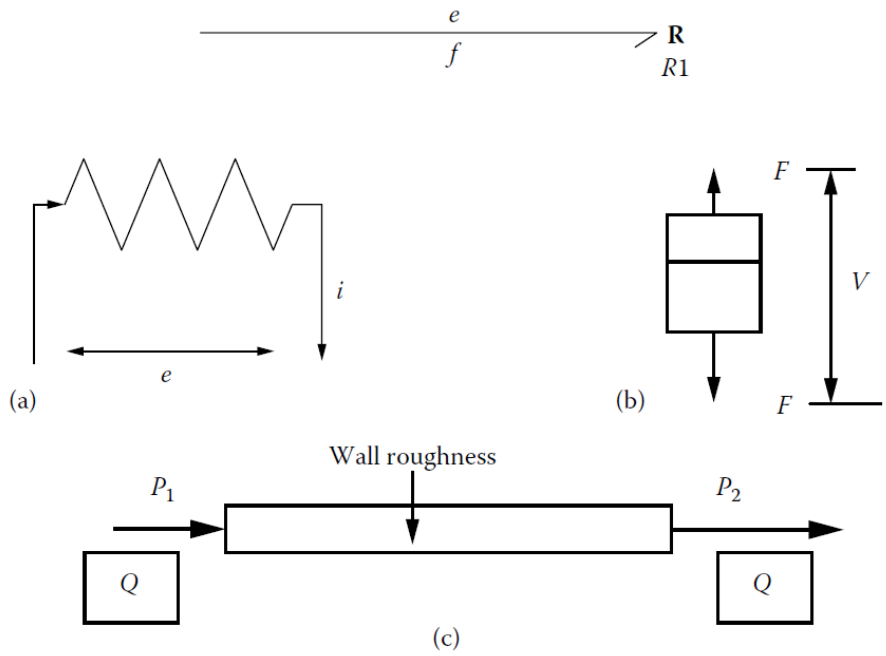


Figure 2.2: Resistive elements: (a) electrical resistance, (b) mechanical damper, and (c) flow through a tube.

is used to describe all types of **energy losses** such as electrical resistance, mechanical friction, or hydraulic flow resistance. The element is represented in the bond graph by the following relation:

The resistive element is defined by a **static** (time-independent) relation linking the effort e and the flow f:

$$\Phi_R(e, f) = 0 \tag{2.9}$$

This means that e is a function of f (or vice versa), without any dynamics or energy storage.

- **Linear case:** $e(t) = R\, f(t)$, where R is a constant resistance or friction coefficient.
- **Nonlinear case:** $e(t) = R[f(t)]$, where the relation between e and f is nonlinear, often used to describe complex or turbulent phenomena.

In different physical domains, the resistive element takes the following forms:

$$\begin{cases} u = R_e i & \text{(Electrical domain — Ohm's law)} \\ F = R_m \dot{x} & \text{(Mechanical domain — viscous friction)} \\ P = R_h Q^2 & \text{(Hydraulic domain — Bernoulli's turbulent flow law)} \end{cases} \tag{2.10}$$

In each case, the **power dissipated** is the product of the effort and flow variables:

$$P_{\text{diss}} = e \cdot f$$

Table 2.4: Examples of resistive phenomena in different physical domains

Domain	Interpretation	Formulation
Translation	Viscous friction	$F = R_t \dot{x}$
	Dry (Coulomb) friction	$F = R_t \, \text{sign}(\dot{x})$
Rotation	Viscous rotational friction	$F = R_r \omega$
	Dry rotational friction	$F = R_r \, \text{sign}(\dot{\omega})$
Hydraulics	Flow limitation (laminar regime)	$P = R_h Q$
	Flow limitation (turbulent regime)	$P = R_h Q^2$
Electricity	Electrical resistance (Ohmic loss)	$U = R_e I$
Thermal	Heat conduction (Fourier's law)	$\dot{S} = \frac{T_1 - T_2}{R_{th}}$
	Thermal contact resistance	$\dot{Q} = \frac{\Delta T}{R_{th}}$
Chemical	Reaction rate limitation	$\mu = R_{ch} \dot{n}$
	Diffusion resistance	$\dot{n} = \frac{\Delta \mu}{R_{ch}}$

This represents the rate at which energy is lost (e.g., converted into heat).
Example interpretation:

- In **electrical systems**, the resistor R_e converts electrical energy into heat according to Ohm's law $U = R_e I$.
- In **mechanical systems**, a viscous damper transforms kinetic energy into heat through fluid friction, modeled as $F = R_t \dot{x}$.
- In **hydraulic systems**, resistance to flow due to pipe geometry or viscosity is represented as $P = R_h Q$ or $P = R_h Q^2$, depending on whether the flow is laminar or turbulent.

Therefore, the resistive element R is essential for capturing **energy dissipation mechanisms** in all physical domains within the bond graph framework.

Capacitive Element (C)

The one-port element C models any system that converts the supplied power into **stored potential energy**. It represents elements that accumulate energy as a function of the **effort variable** e and the **time integral of the flow variable** f. Figure 2.3 shows some examples of these types of devices in different domains.

The generic relation defining this element is a dynamic relationship linking the integral of the flow (the generalized displacement) and the effort:

$$\Phi_C(e, \int f(t)dt) = \Phi_C(e, q) = 0 \tag{2.11}$$

In the linear case, we have:

$$q(t) = C\, e(t), \quad \text{with} \quad q(t) = \int f(t)dt \tag{2.12}$$

which can also be written as:

$$e(t) = \frac{1}{C} \int_0^t f(\tau)d\tau \quad \text{or} \quad f(t) = C\frac{de}{dt} \tag{2.13}$$

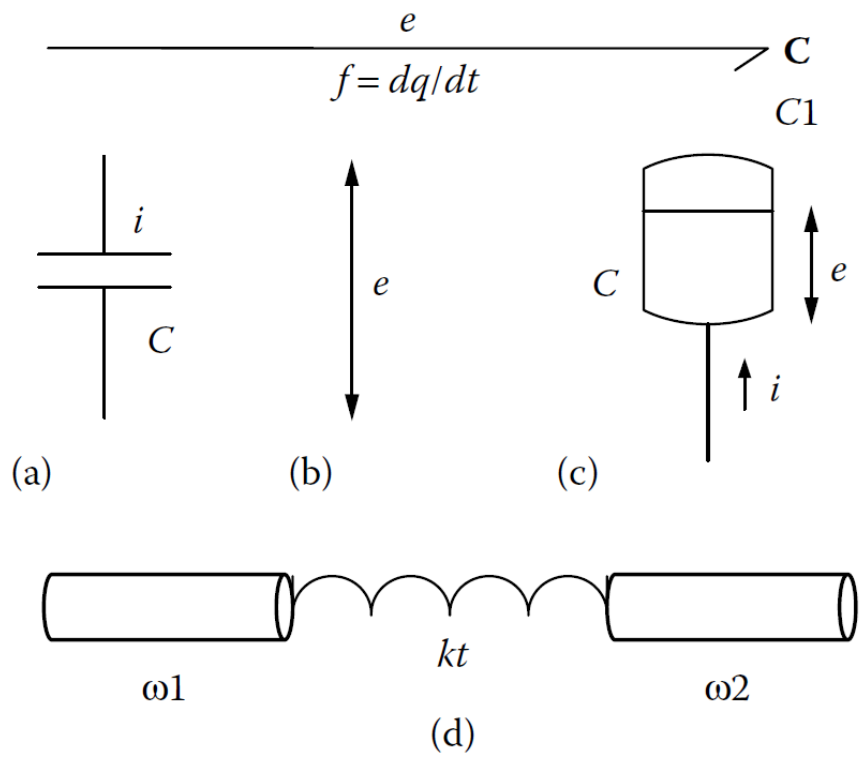


Figure 2.3: 1-port capacitive elements: (a) linear electric capacitor, (b) mechanical spring, (c) fluid stored at some height or accumulator, and (d) torsional spring.

Table 2.5: Examples of capacitive phenomena in different physical domains

Domain	**Interpretation**	**Formulation**
Translation	Linear spring	$\dot{x} = \frac{1}{k}\frac{dF}{dt} \Rightarrow F = k\int_0^t \dot{x}(\tau)d\tau$
Rotation	Torsional spring	$\omega = \frac{1}{k}\frac{d\Gamma}{dt} \Rightarrow \Gamma = k\int_0^t \omega(\tau)d\tau$
Hydraulics	Reservoir (fluid compliance)	$Q = C_h\frac{dP}{dt} \Rightarrow P = \frac{1}{C_h}\int_0^t Q(\tau)d\tau$
Electricity	Capacitor	$I = C_e\frac{dU}{dt} \Rightarrow U = \frac{1}{C_e}\int_0^t I(\tau)d\tau$
Thermal	Thermal capacity (heat storage)	$\dot{Q} = C_t\frac{dT}{dt} \Rightarrow T = \frac{1}{C_t}\int_0^t \dot{Q}(\tau)d\tau$
Chemical	Chemical accumulation (mass or concentration storage)	$\dot{n} = C_{ch}\frac{d\mu}{dt} \Rightarrow \mu = \frac{1}{C_{ch}}\int_0^t \dot{n}(\tau)d\tau$

Examples of Capacitive Elements

Stored Energy Expression The capacitive element stores (potential energy), expressed as:

$$E(t) = \int_0^t e(\tau)\, C\frac{de}{d\tau}\, d\tau = \frac{1}{2}Ce^2(t) = \frac{1}{2}\frac{q^2(t)}{C} \tag{2.14}$$

$$\begin{aligned} &\text{Translation:} \quad E(t) = \tfrac{1}{2}kx^2(t) \\ &\text{Electricity:} \quad E(t) = \tfrac{1}{2}\tfrac{q^2(t)}{C} \end{aligned} \tag{2.15}$$

Thermal Domain Interpretation To define the element C in the thermal domain, consider a body heated by a heat flow $\dot{Q}$ (thermal power). Body temperature T depends on the amount of heat energy accumulated:

$$Q = \int_0^t \dot{Q}(t)dt + Q_0 \tag{2.16}$$

Assuming that the body's total thermal capacity C_t remains constant, the temperature variation is given by:

$$T = T_0 + \frac{Q}{C_t} \tag{2.17}$$

Thus, the **thermal capacitance** stores energy in the form of temperature rise.

Inertial Element (I)

The element I (inertia) is also an energy storage element. It transforms the supplied power into stored energy in different physical domains:

- Kinetic energy for mechanical systems.
- Magnetic energy for electrical systems.

Figure 2.4 shows some examples of these types of devices in different domains. The inertial element is defined by a dynamic relation linking the *flow* $f(t)$ and the *momentum* $p(t)$:

$$p(t) = If(t), \quad \text{with} \quad p(t) = \int_0^t e(\tau)d\tau \tag{2.18}$$

or equivalently,

$$f(t) = \frac{1}{I}\int_0^t e(\tau)d\tau \quad \text{and} \quad e(t) = I\frac{df}{dt} \tag{2.19}$$

In the linear case, the general constitutive laws are expressed as:

- Electrical domain: $u = L\frac{di}{dt}$ (magnetic energy storage in an inductor)
- Mechanical translation: $F = m\frac{d\dot{x}}{dt}$ (kinetic energy in a mass)
- Mechanical rotation: $\Gamma = J\frac{d\omega}{dt}$ (rotational inertia)

More generally, in other physical domains, the inertial element characterizes the delay or accumulation of effort caused by the change in flow. For example:

- In thermal systems, thermal inertia relates the rate of temperature change to the applied heat flux.
- In chemical systems, it can describe the delay between the chemical potential and the rate of reaction.

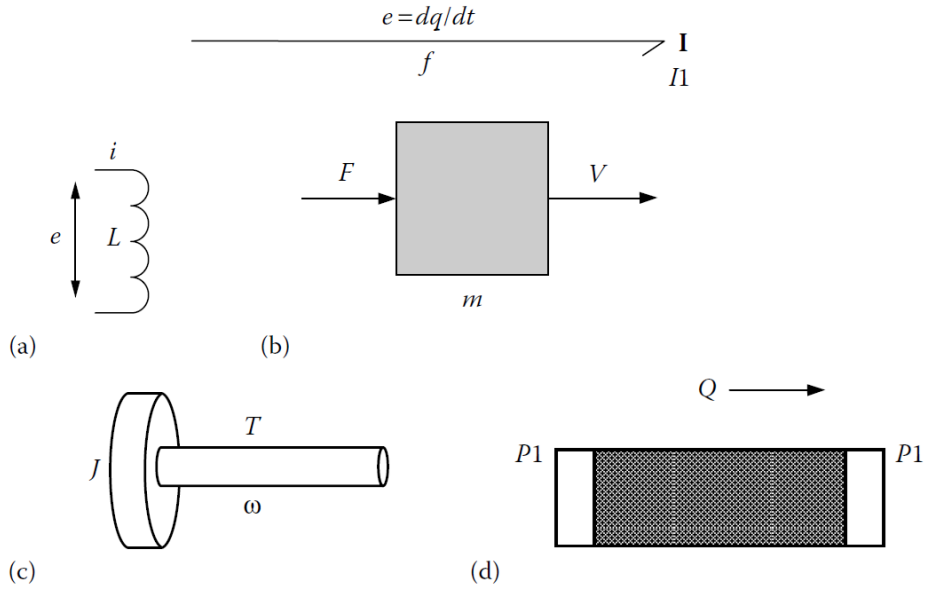


Figure 2.4: 1-port inductive elements: (a) electrical inductor, (b) mechanical mass, (c) rotational inertia, and (d) fluid inertia.

Table 2.6: Examples of inertial phenomena in different physical domains

Domain	**Interpretation**	**Formulation**
Translation	Translational mass	$F = m\frac{d\dot{x}}{dt} \Rightarrow \dot{x} = \frac{1}{m}\int_0^t F(\tau)d\tau$
Rotation	Rotational inertia	$\Gamma = J\frac{d\omega}{dt} \Rightarrow \omega = \frac{1}{J}\int_0^t \Gamma(\tau)d\tau$
Hydraulics	Fluid column (inertial effect)	$P = I_h\frac{dQ}{dt} \Rightarrow Q = \frac{1}{I_h}\int_0^t P(\tau)d\tau$
Electricity	Inductance	$U = L\frac{dI}{dt} \Rightarrow I = \frac{1}{L}\int_0^t U(\tau)d\tau$
Chemical	Reaction inertia	$\mu = I_c\frac{dr}{dt} \Rightarrow r = \frac{1}{I_c}\int_0^t \mu(\tau)d\tau$

Examples of Inertial Elements: The inertial element stores **kinetic or magnetic energy**, depending on the domain, as:

$$E(t) = \int_0^t e(t)f(t)dt = \int_0^t I\frac{df}{dt}f(t)dt = \frac{1}{2}If^2(t) \tag{2.20}$$

Examples:

- Mechanical translation: $E(t) = \frac{1}{2}m\dot{x}^2(t)$
- Electrical: $E(t) = \frac{1}{2}Li^2(t)$
- Chemical: $E(t) = \frac{1}{2}I_c r^2(t)$

Thus, the inertial element I represents the accumulation of energy due to motion, current, or reaction rate, depending on the physical domain. It is an essential component in dynamic modeling because it introduces the concept of **memory** or **delay** — the effect of effort is not instantaneous but depends on the time derivative of the flow.

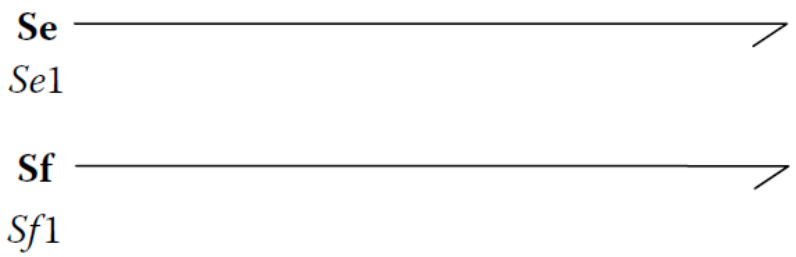


Figure 2.5: Bond graph presentation

2.4.2 Active Elements

Active elements are components that **supply power** to the system. They represent external energy sources that impose either an *effort* or a *flow* on the connected system.

There are two types of active elements:

- **Effort Source (Se)** — imposes an effort variable (e.g., voltage, force, pressure, temperature difference).
- **Flow Source (Sf)** — imposes a flow variable (e.g., current, velocity, flow rate, heat flux).

Example

A pressure pump that applies a constant pressure P to a hydraulic circuit is an example of an **effort source (Se)**. This pressure causes a volumetric flow rate Q to circulate in the system. The product $P \times Q$ represents the instantaneous power delivered by the source.

In general, the instantaneous power supplied by an active element is given by:

$$P(t) = e(t) \cdot f(t) \tag{2.21}$$

where $e(t)$ is the imposed effort or flow variable depending on the source type.

Physical Interpretation:

- **Effort Source (Se)** — behaves like a fixed potential (voltage, force, pressure) that drives the system regardless of the resulting flow.
- **Flow Source (Sf)** — behaves like a constant motion or flow generator that determines the rate of transfer (velocity, current, heat flux, etc.).

Examples of Active Elements in Different Physical Domains:

Explanation: Effort and flow sources represent the external interactions between the modeled system and its environment:

- In **mechanical systems**, an effort source applies a force or torque regardless of motion, while a flow source imposes a specific velocity or angular speed.

Table 2.7: Effort Sources (Se) in Various Physical Domains

Domain	Effort Source (Se)
Electrical	Voltage source: U = constant
Translational Mechanics	Force source: F = constant
Rotational Mechanics	Torque source: Γ = constant
Hydraulics	Pressure source: P = constant
Thermal	Temperature source: T = constant
Chemical	Chemical potential source: μ = constant

Table 2.8: Flow Sources (Sf) in Various Physical Domains

Domain	Flow Source (Sf)
Electrical	Current source: I = constant
Translational Mechanics	Velocity source: $\dot{x}$ = constant
Rotational Mechanics	Angular velocity source: ω = constant
Hydraulics	Flow source: Q = constant
Thermal	Heat flux source: $\dot{Q}$ = constant
Chemical	Reaction rate source: r = constant

- In **electrical systems**, voltage and current sources define the input power delivered to the circuit.
- In **hydraulic systems**, pumps act as effort sources (constant pressure) or flow sources (constant flow rate).
- In **thermal systems**, a heater imposing a fixed temperature corresponds to an effort source, while one imposing constant heat flux corresponds to a flow source.
- In **chemical systems**, a source may fix the chemical potential (effort) or the reaction rate (flow).

Thus, active elements serve as the **driving energy inputs** of any bond graph model, defining how external power is injected into the system.

2.4.3 Junction Elements

The junction elements, denoted as (0, 1, TF, and GY), are used to connect the passive and active elements (R, I, C, Se, Sf) in order to construct the system architecture to be modeled. All these junctions are **power-conserving**, meaning that the total power entering and leaving the junction is always zero.

0-Junction and 1-Junction

To reproduce the structure of a physical system, the elements R, C, and I are interconnected using:

- **0-junctions (effort junctions)** — used when elements share the same effort variable.

- **1-junctions (flow junctions)** — used when elements share the same flow variable.

For example:

- In **mechanics**, a 0-junction corresponds to elements in series (same force), while a 1-junction corresponds to elements in parallel (same velocity).
- In **electrical systems**, a 0-junction corresponds to parallel connections (same voltage), and a 1-junction corresponds to series connections (same current).
- In **hydraulics**, 0-junctions correspond to equal pressure points, and 1-junctions correspond to equal flow rates.

For the 0-Junction: The defining equations for a 0-junction (common effort) are:

$$\begin{cases} e_1 = e_2 = e_3 = e_4, & \text{(equality of efforts)} \\ f_1 + f_2 + f_3 + f_4 = 0, & \text{(sum of flows is zero)} \\ e_1 f_1 + e_2 f_2 + e_3 f_3 + e_4 f_4 = 0, & \text{(power balance)} \end{cases} \tag{2.22}$$

For the 1-Junction: The defining equations for a 1-junction (common flow) are:

$$\begin{cases} f_1 = f_2 = f_3 = f_4, & \text{(equality of flows)} \\ e_1 + e_2 + e_3 + e_4 = 0, & \text{(sum of efforts is zero)} \\ e_1 f_1 + e_2 f_2 + e_3 f_3 + e_4 f_4 = 0, & \text{(power balance)} \end{cases} \tag{2.23}$$

These two junction types are essential to ensure the conservation of power and the consistency of the physical laws across domains.

Transformer (TF) and Gyrator (GY)

Power transformation phenomena between two energy domains are represented by two-port elements:

- **TF (Transformer):** Converts effort and flow between ports using a proportionality constant n, such that

$$e_1 = n e_2, \quad f_2 = n f_1 \tag{2.24}$$

  Examples include gear trains in mechanics, or transformers in electrical systems.

- **GY (Gyrator):** Couples effort and flow in a crossed manner, such that

$$e_1 = r f_2, \quad e_2 = r f_1 \tag{2.25}$$

  Examples include electromechanical devices like DC motors, where torque and current are interrelated.

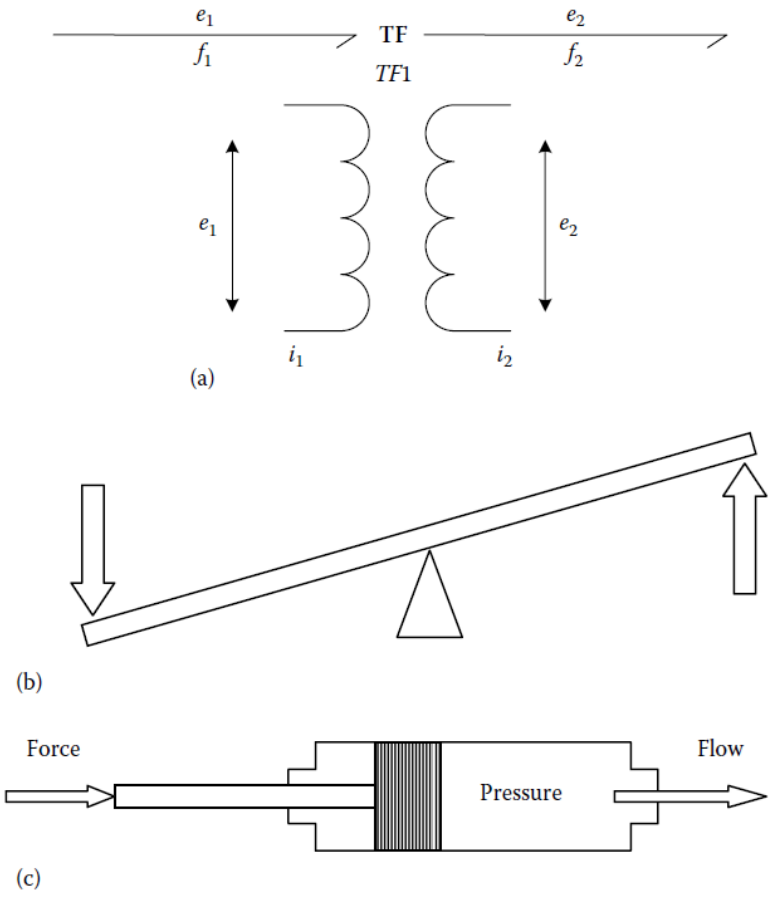


Figure 2.6: Schematics showing transformer elements within different domains: (a) electrical trans-former, (b) mechanical transformer (lever), and (c) hydraulic transformer (plunger).

2.4.4 Information Links

When no power transfer occurs in a part of the model—or when one of the power variables (effort or flow) is negligible—a special connection called an **information link** is used. This is typically the case for sensors or measurement elements.

Sensors (also called **detectors**) do not exchange power with the system; they only transmit information:

- **Df:** Flow detector — measures the flow variable (e.g., velocity sensor, current sensor).
- **De:** Effort detector — measures the effort variable (e.g., force sensor, voltage sensor).

These information links are represented by dashed lines in bond graph diagrams.

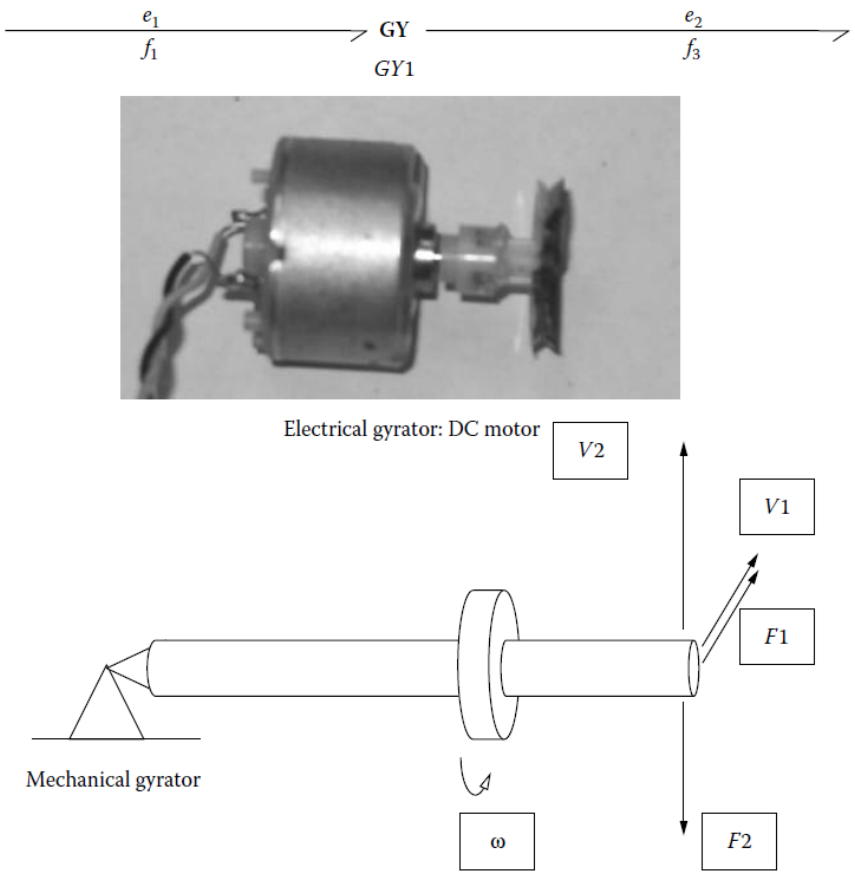


Figure 2.7: Examples of gyrator elements.

2.5 Conclusion

In this chapter, we are presented with the overall elements of the bond graph language, including the basic components such as resistive (R), capacitive (C), and inertial (I) elements, as well as the junctions (0 and 1) and power transformers (TF and GY). The chapter also discussed active sources of effort (Se) and flow (Sf), and the rules for constructing and simplifying bond graph models. Through various examples across mechanical, electrical, hydraulic, thermal, and chemical domains, we illustrated how bond graphs provide a unified, systematic approach to modeling dynamic systems, allowing both energy storage and dissipation to be represented clearly. This framework lays the foundation for more advanced modeling, simulation, and control of complex physical systems.

CHAPTER 3

Bond Graph Model Construction

3.1 Introduction

In this chapter, we discuss the methods and techniques used to construct the Bond Graph model of a physical (dynamic) system.
Before moving on to the development techniques of systems using Bond Graphs, we must first understand some commonly used simplification rules that will help us draw simplified representations of the Bond Graph.

3.2 Simplification Rules for Junction Structures

The simplification rules are applied to the basic Bond Graph structure to make the system more compact. There are six simplification rules.

Rule 01: These two representations are equivalent. The first can be replaced by the second. If a single junction connects only two bonds (links), then the effort coming from the left side of the 0-junction is the same effort leaving the right side of the junction. Similarly, the flow coming from the left side is the same as the one going to the right.

Remark. This simplification does not work if a third bond is connected to the 0-junction.

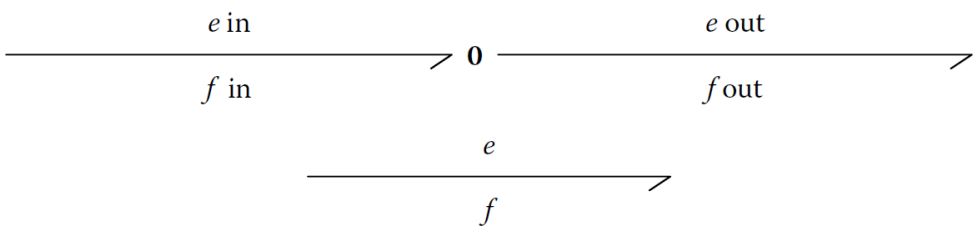


Figure 3.1: Simplification rule.

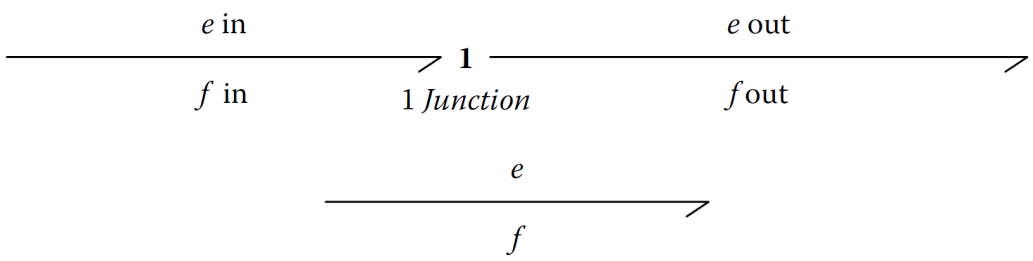


Figure 3.2: Simplification rule.

Rule 02: From the diagram above, we can deduce that:

$$\begin{cases} e_{en} - e_s = 0 & \Rightarrow \quad e_{en} = e_s \\ P_{en} - P_s = 0 & \Rightarrow \quad e_{en} f_{en} = e_s f_s \\ f_{en} = f_s \end{cases} \tag{3.1}$$

Therefore, the two representations are equivalent.

Remark. This simplification does not work if a third bond is connected to the 1-junction.

Rule 03: The 0-junction is an equal-effort junction. Therefore, $e_1 = e_2 = e_3 = e_4 = e_5$ (in representation (a)). The flow splits at the 0-junction. Thus, in the first representation:

$$\begin{cases} f_1 = f_2 + f_3 \\ f_3 = f_4 + f_5 \end{cases} \Rightarrow f_1 = f_2 + f_4 + f_5 \tag{3.2}$$

which is equivalent to the second representation.

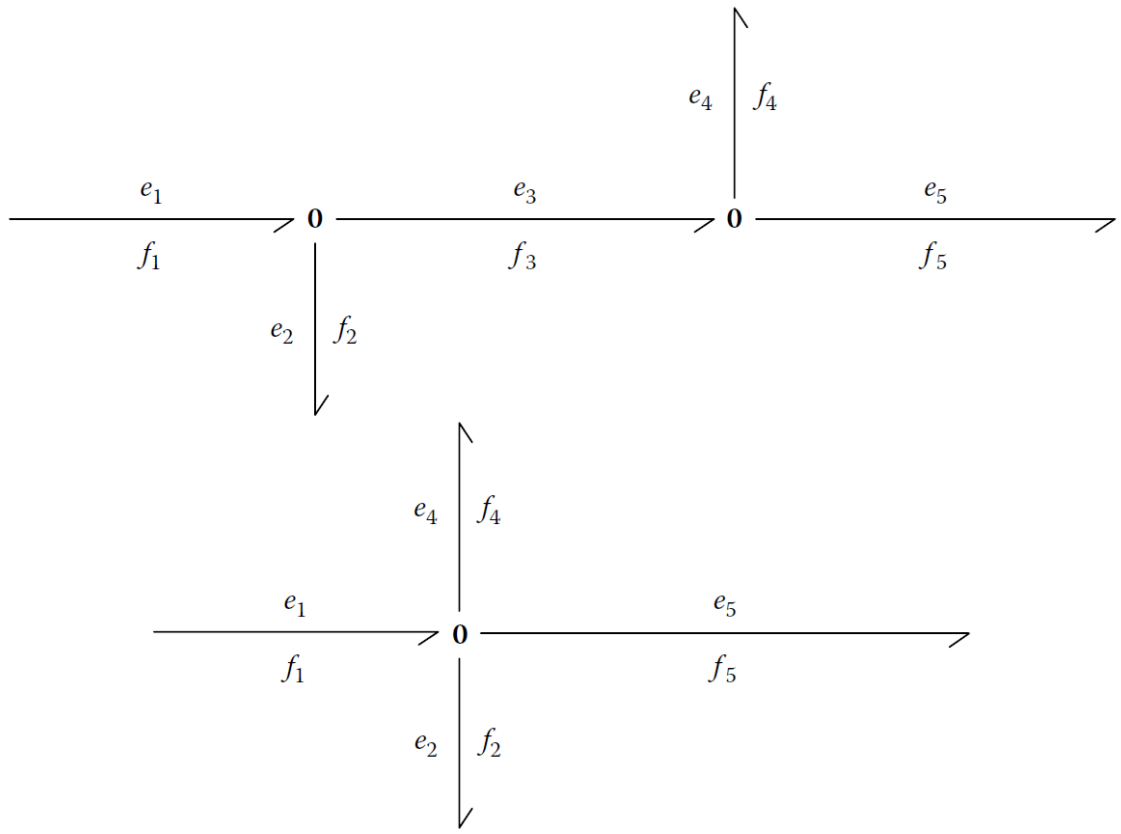


Figure 3.3: Simplification rule.

Rule 04: These two representations are equivalent. The 1-junction is an equal-flow junction. Therefore, $f_1 = f_2 = f_3 = f_4 = f_5$. In the first representation, the effort splits at the 0-junction. Thus, in the first representation:

$$\begin{cases} e_1 = e_2 + e_3 \\ e_3 = e_4 + e_5 \end{cases} \Rightarrow e_1 = e_2 + e_4 + e_5 \tag{3.3}$$

which corresponds to the second representation.

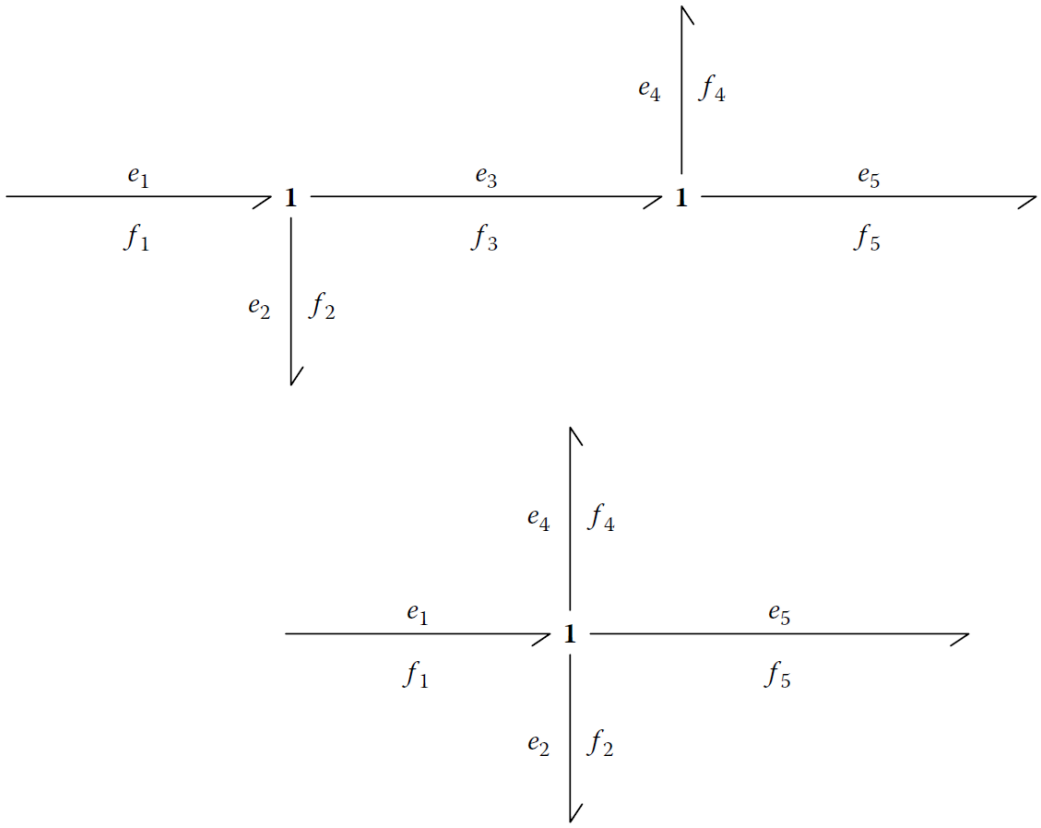


Figure 3.4: Simplification rule.

Rule 05:

First Bond Graph Figure (1.a) contains four junctions, labeled 1, 2, 3, and 4, as shown:

At junction 1:

$$\begin{cases} e_1 = e_a = f_3 \\ f_1 = f_a + f_b \end{cases} \tag{3.4}$$

At junction 2:

$$\begin{cases} e_a = e_2 + e_c \\ f_a = f_2 = f_c \end{cases} \tag{3.5}$$

At junction 3:

$$\begin{cases} e_b = e_3 + e_d \\ f_b = f_3 = f_d \end{cases} \tag{3.6}$$

At junction 4:

$$\begin{cases} e_c = e_d = e_4 \\ f_c + f_d = f_4 \end{cases} \tag{3.7}$$

Combining the flow equations of junctions 1 and 2, we get:

$$f_1 = f_2 + f_b \tag{3.8}$$

From equation (3.6) of junction 3 ($f_b = f_3$), we obtain:

$$f_1 = f_2 + f_3 \tag{3.9}$$

From the flow equation (3.8) of junction 4, since ($f_c = f_2$ and $f_d = f_3$), we can write:

$$f_4 = f_2 + f_3 \tag{3.10}$$

Combining the two flow equations (3.10) and (3.11), we get:

$$f_1 = f_2 + f_3 = f_4 \tag{3.11}$$

Combining the effort equations (3.1) and (3.2) of junctions 1 and 2, we have:

$$e_1 = e_2 + e_c \tag{3.12}$$

From junction 4 (equation (3.7)), we know $e_c = e_4$. Therefore:

$$e_1 = e_2 + e_4 \tag{3.13}$$

Combining the effort equations (3.1) and (3.5) of junctions 1 and 3, we obtain:

$$e_1 = e_3 + e_4 \tag{3.14}$$

Therefore, from (3.14) and (3.16), we get:

$$e_1 = e_3 + e_4 = e_2 + e_4 \tag{3.15}$$

Second Bond Graph At junction 1:

$$\begin{cases} e_1 = e^* + e_4 \\ f_1 = f_4 = f^* \end{cases} \tag{3.16}$$

At junction 0:

$$\begin{cases} e_3 = e_2 + e^* \\ f^* = f_2 = f_3 \end{cases} \tag{3.17}$$

Combining equations (3.18) with (3.20) and (3.19) with (3.21), we obtain:

$$\begin{cases} e_1 = e_2 + e_4 \\ f_1 = f_4 = f_2 + f_3 \end{cases} \tag{3.18}$$

By comparison, we can see that the two representations are equivalent.

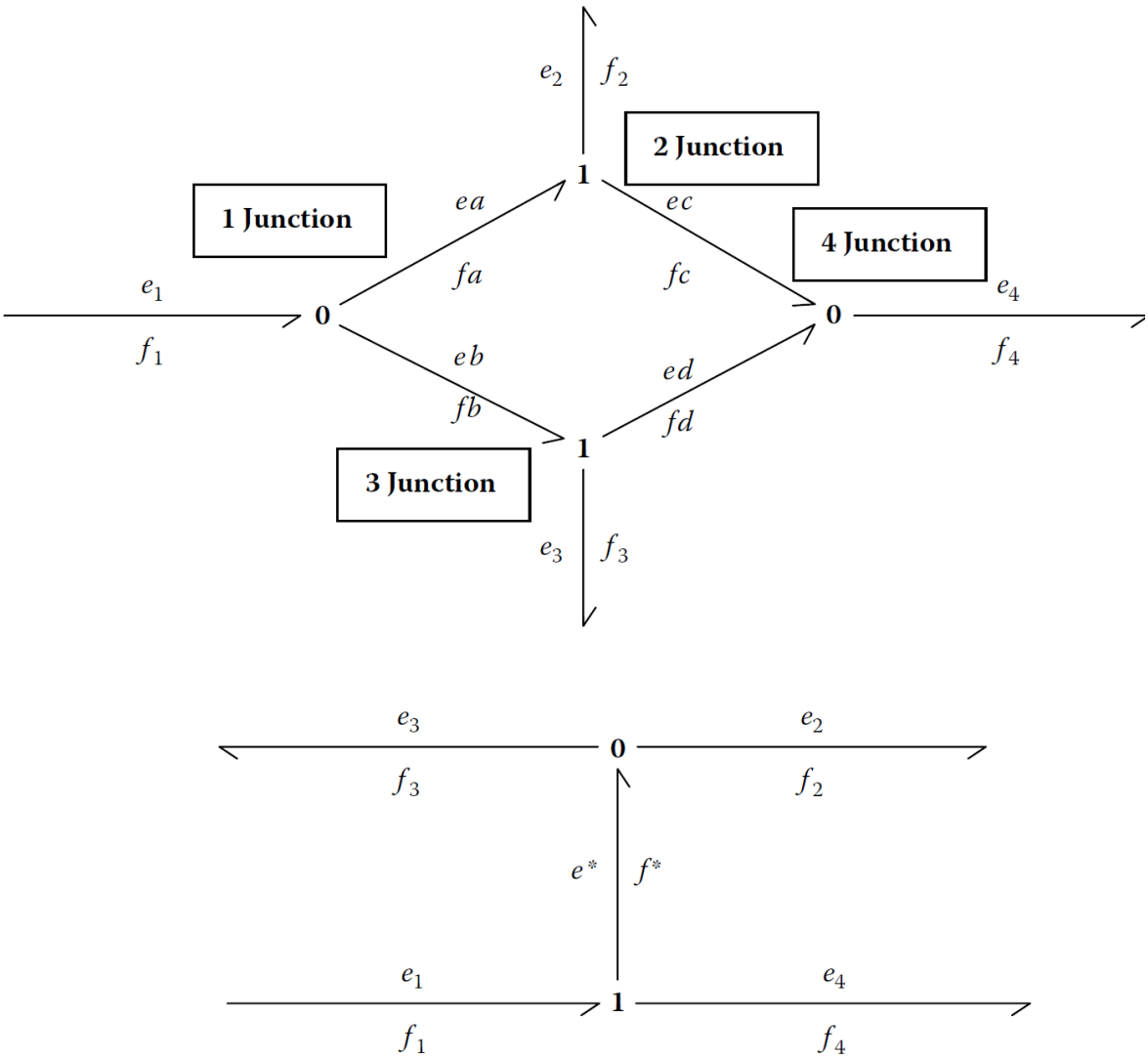


Figure 3.5: Simplification rule.

Rule 06: We have discussed the different simplification rules. Now, we move on to the process of Bond Graph development and representation for dynamic systems.

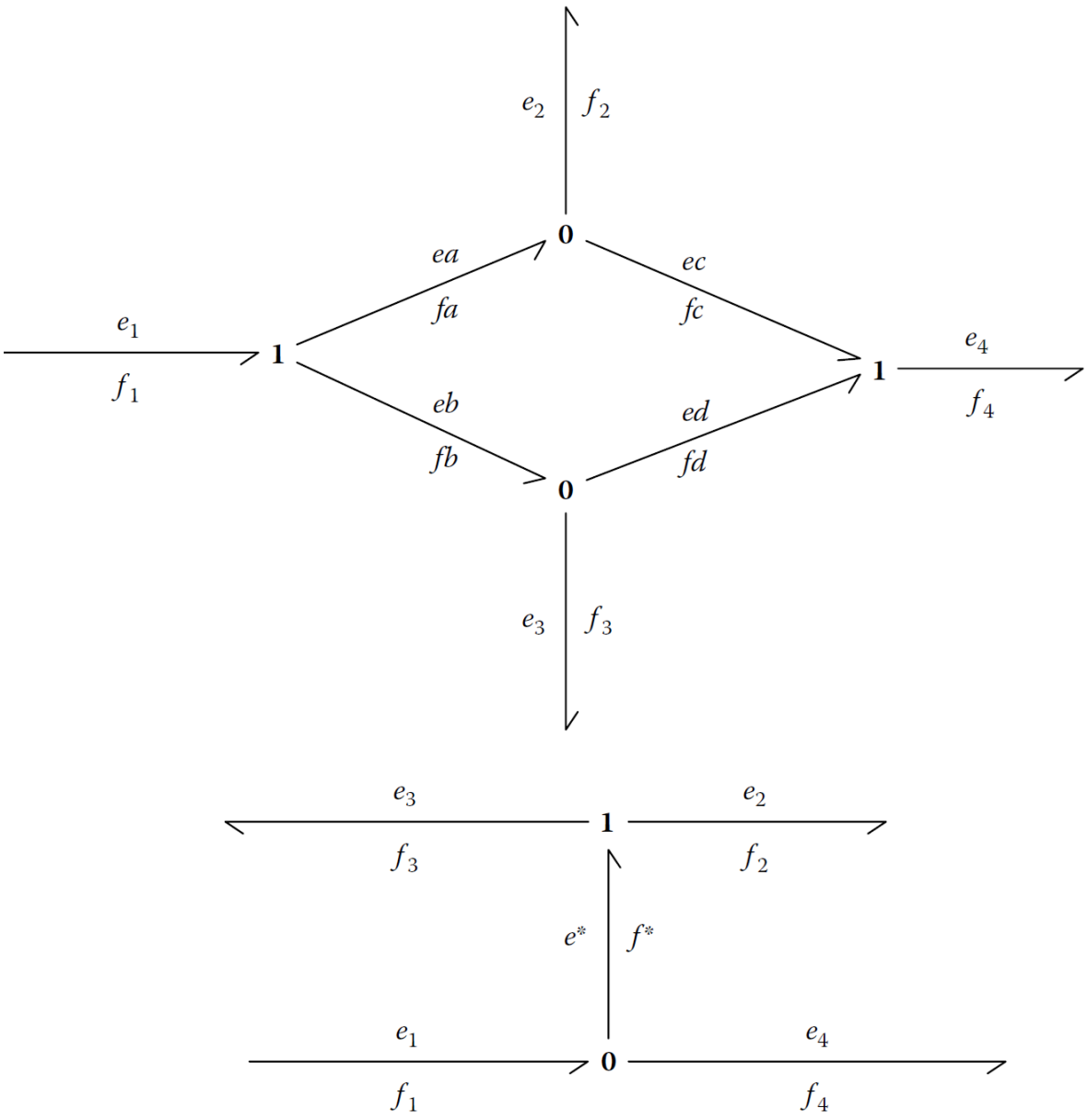


Figure 3.6: Simplification rule.

We will now move on to the process of developing and representing Bond Graphs for dynamic systems.

3.3 Bond Graphs for Electrical Systems

The construction of a Bond Graph can be approached by considering the following steps:

1. Assign a 0-junction to each node with a distinct potential.

2. Each single-port circuit element is connected to a 1-junction via a power bond and positioned between the appropriate 0-junctions.

3. Assign power direction to all bonds in the model.

4. Remove the 0-junction and its connected bonds at each known ground potential. If no ground is specified, choose one 0-junction as the reference and remove it with all attached bonds.

5. Simplify the bond graphs by using all rules of simplification.

Example: Figur 3.7 shows a circuit with points a, b, c, and d representing four distinct potentials. The bond graph is developed step by step. After the first three steps, Figure 3.8 is obtained. Junction d represents the ground and is removed (Step 4), resulting in the simpler bond graph shown in Figure 3.9. Further simplification is achieved by removing redundant junctions connected to a single element, leading to the final diagram in Figure 3.10.

The vertical lines at the end of each bond are called *causal strokes*. A detailed discussion on causality and causal strokes is provided later.

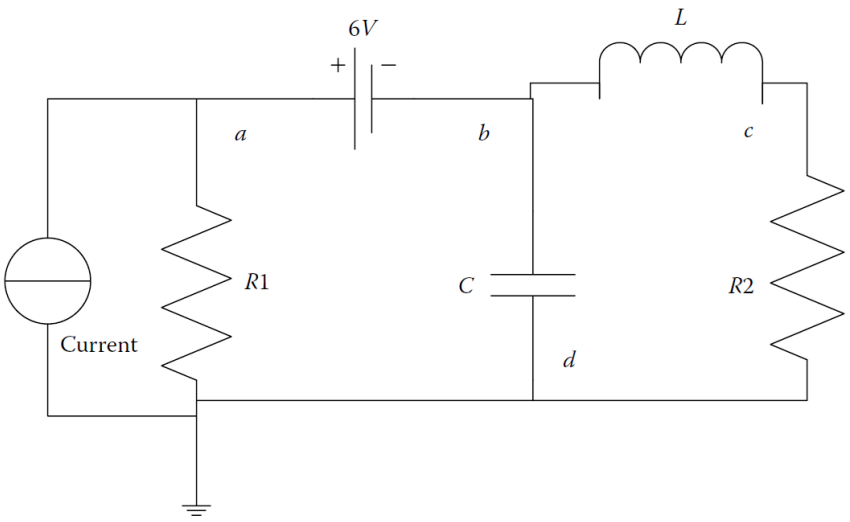


Figure 3.7: The electrical circuit

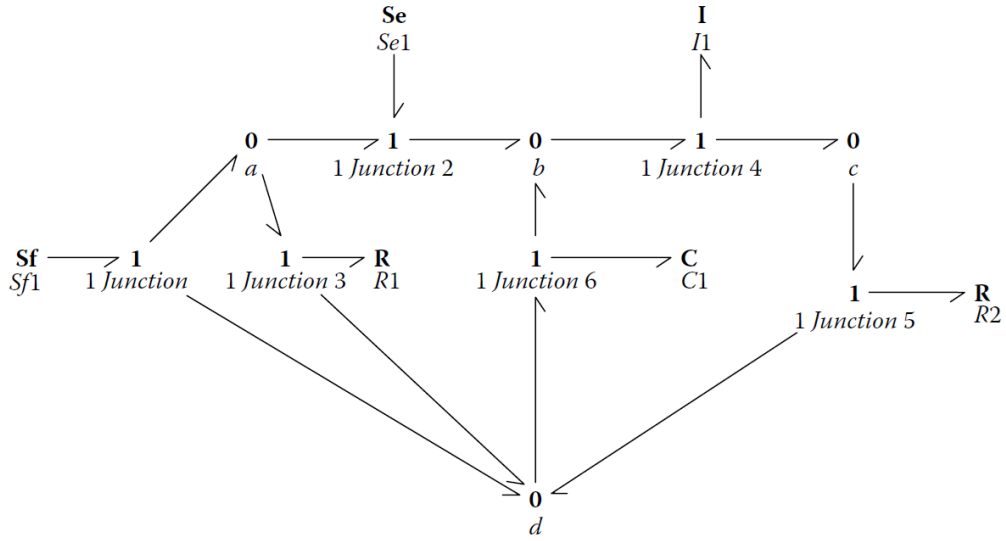


Figure 3.8: Initial bond graph

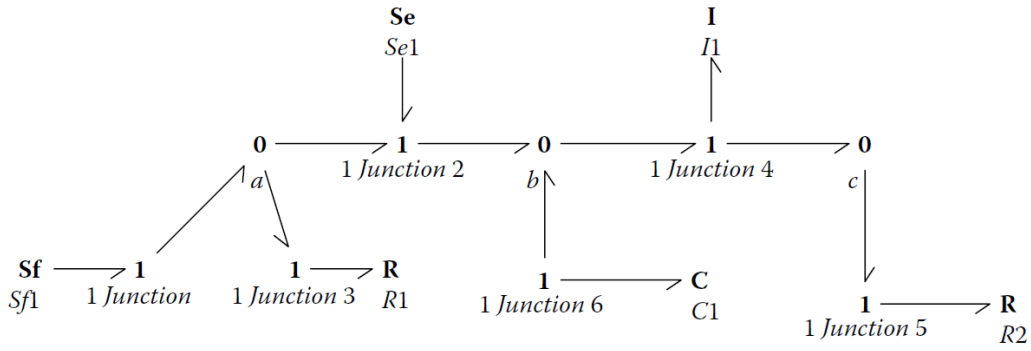


Figure 3.9: Simplified model

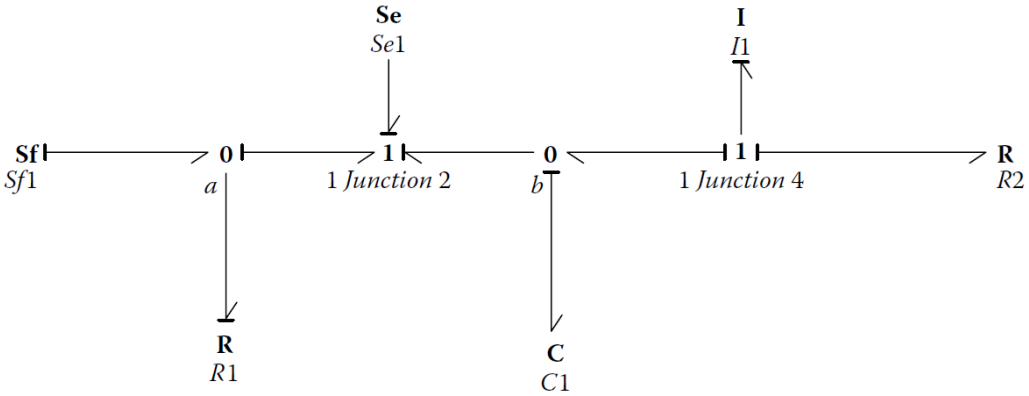


Figure 3.10: Final bond graph

Example 02: Consider the Wheatstone bridge circuit in Figure 3.11. The resistors are labeled R_1 through R_5, and the four potential points are denoted as A, B, C, and D. Following the procedure in Example 01 , the steps are executed sequentially.

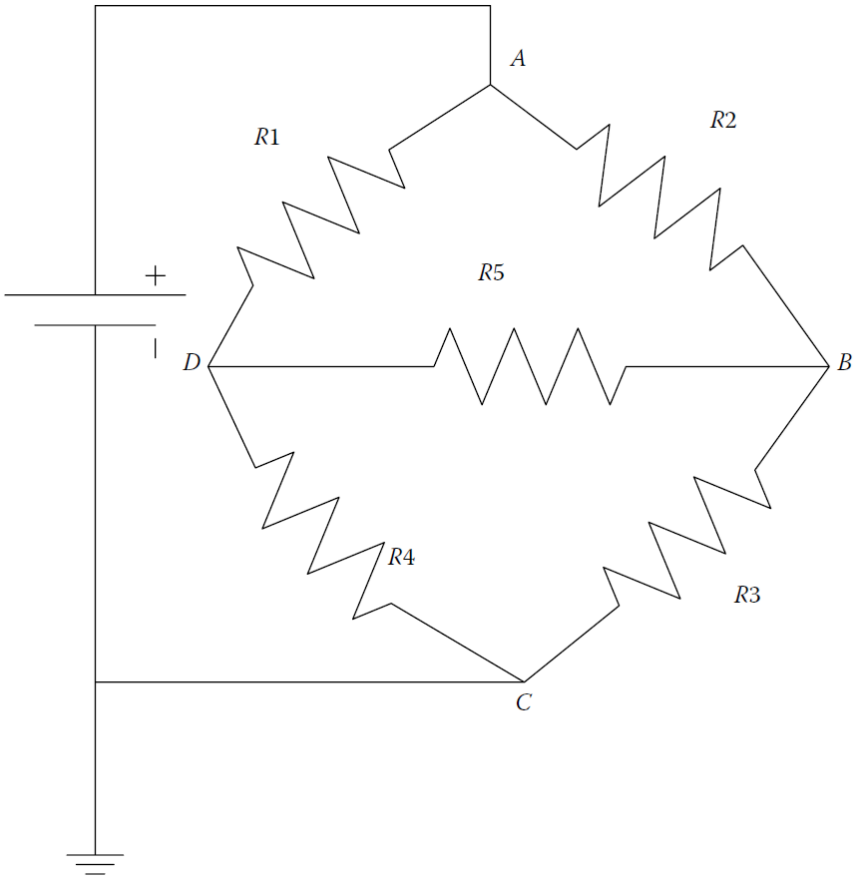


Figure 3.11: Wheatstone bridge Model

After the first three steps, the model in Figure 3.12 is obtained. According to Rule 04, the bonds connected to the ground potential are then removed, resulting in the bond graph shown in Figure 3.13.
After applying the simplification steps, the bond graph representation becomes as shown in Figure 3.14.

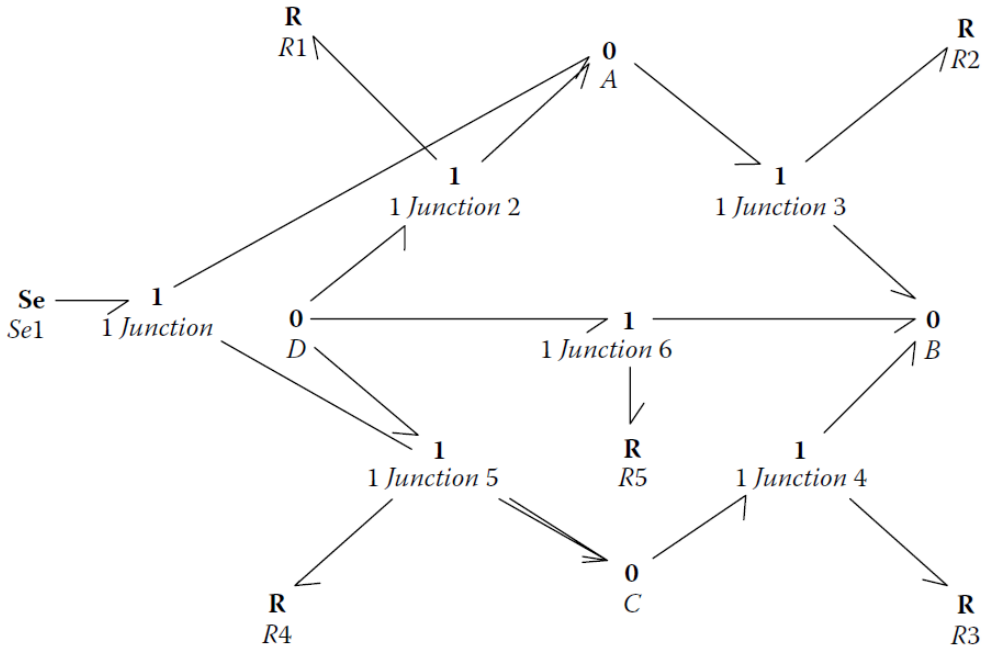


Figure 3.12: Initial bond graph

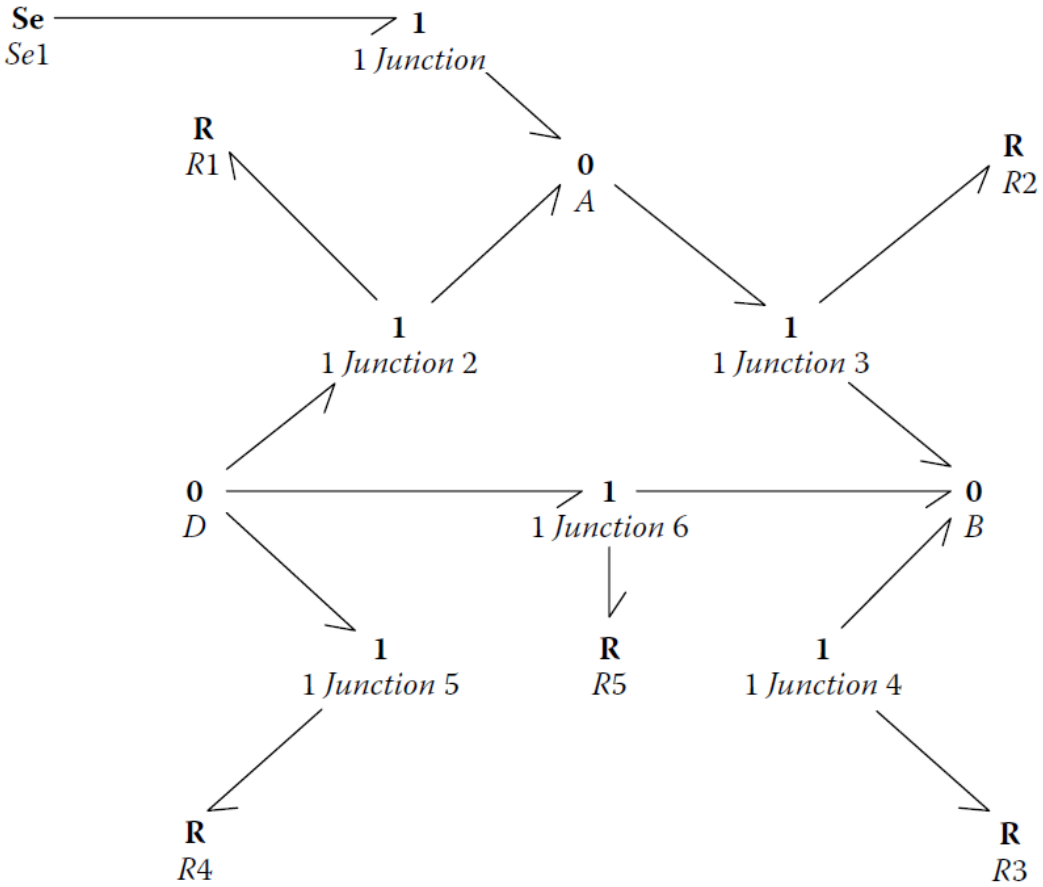


Figure 3.13: Simplified ed model

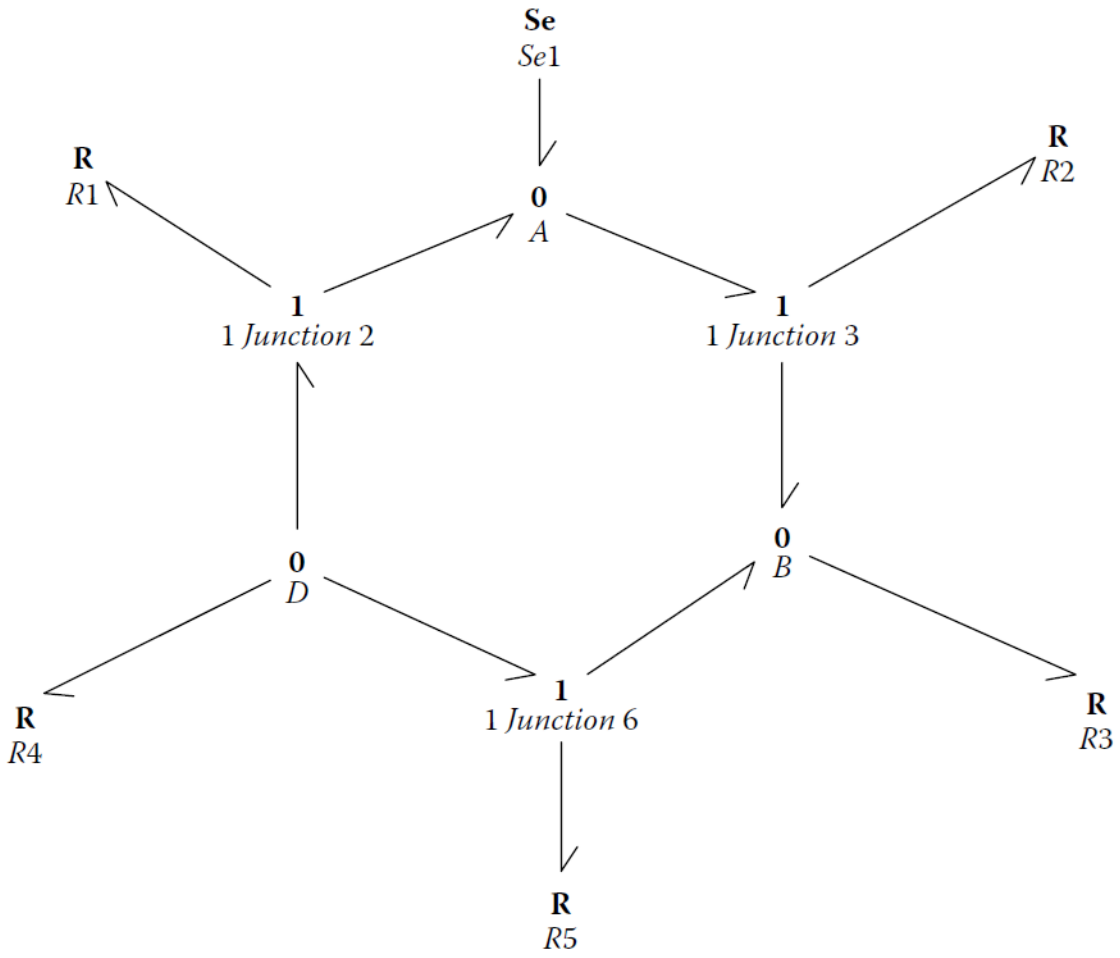


Figure 3.14: Final bond graph model

3.4 Bond Graphs for Mechanical Systems

The following steps can be used to construct the bond graph of a mechanical system:

1. **Identify the system elements:** Determine all mechanical components such as masses, springs, dampers, and actuators.
2. **Assign effort and flow variables:** For translational mechanics, effort corresponds to force F and flow corresponds to velocity $\dot{x}$. For rotational mechanics, effort corresponds to torque Γ and flow corresponds to angular velocity ω.
3. **Represent storage and dissipative elements:**
   - Inertia elements (I) represent masses or rotational inertia.
   - Compliance elements (C) represent springs.
   - Resistive elements (R) represent dampers or friction.
4. **Identify sources:** Determine effort sources (Se) and flow sources (Sf) such as external forces or velocities applied to the system.
5. **Connect elements using junctions:**

- Use **0-junctions** for elements sharing the same effort (force/torque).
- Use **1-junctions** for elements sharing the same flow (velocity/angular velocity).

6. **Incorporate transformers (TF) and gyrators (GY) if needed:** These represent power conversion between domains (e.g., translational to rotational motion).
7. **Simplify the bond graph:** Remove redundant junctions and apply simplification rules to obtain a compact representation.

Example 01: Consider the mechanical system shown in Figure 3.15, which consists of a spring, a mass, and a damper, with an external force applied. To draw its bond graph representation, define two velocities, v_1 and v_2 (with $v_2 = 0$). Represent these velocities using 1-junctions. For velocity v_1, a flow source is added. The initial bond graph is illustrated in Figure 3.16.
By simplifying the model and removing 0-junctions, the bond graph becomes as shown in Figure 3.17. Further simplification yields the final bond graph, presented in Figure 3.18.

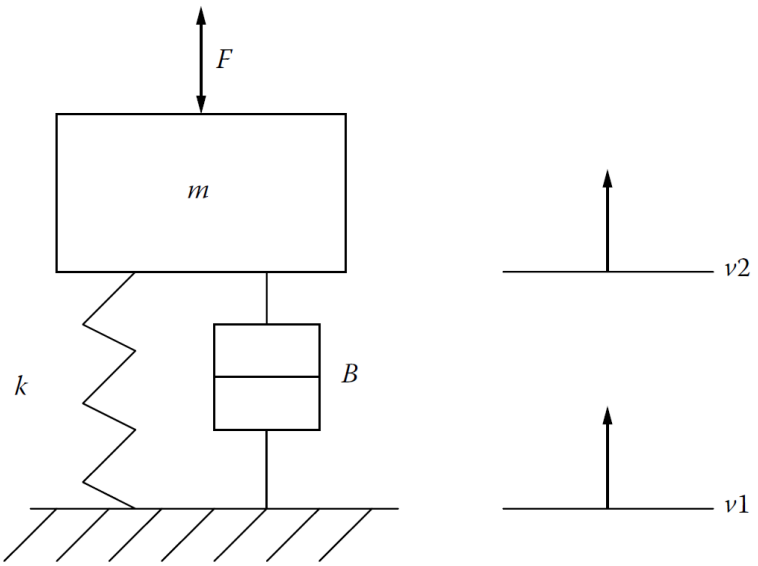


Figure 3.15: Mechanical model

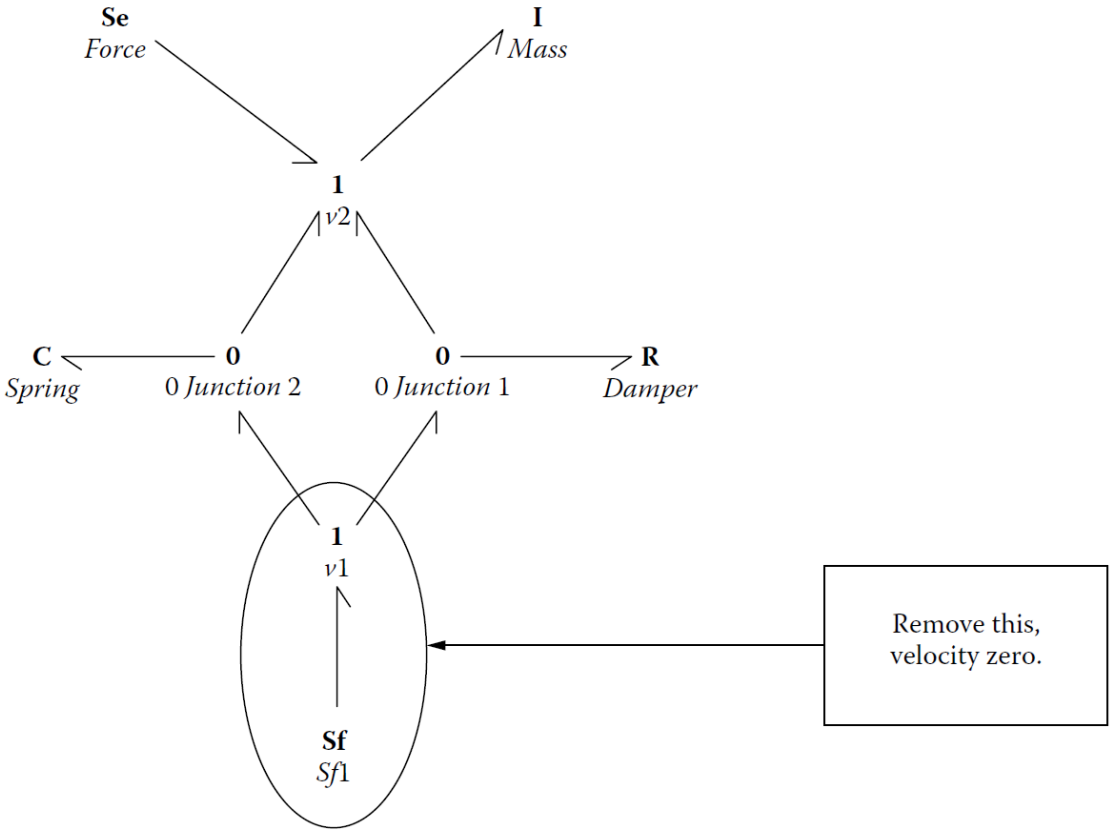


Figure 3.16: Initial bond graph representation

Example 02: The Figure 3.19 represents a two-degree-of-freedom translational mechanical system consisting of:

- M_1 and M_2: two moving masses.
- k_1: a spring connecting M_1 and M_2.
- B_3: a damper in parallel with the spring.
- F: an external force acting on mass M_2.
- B_1 and V_3: represent the contact/friction (damping) between each mass and the ground.
- V_1, V_2, V_3: velocities of each element or node.

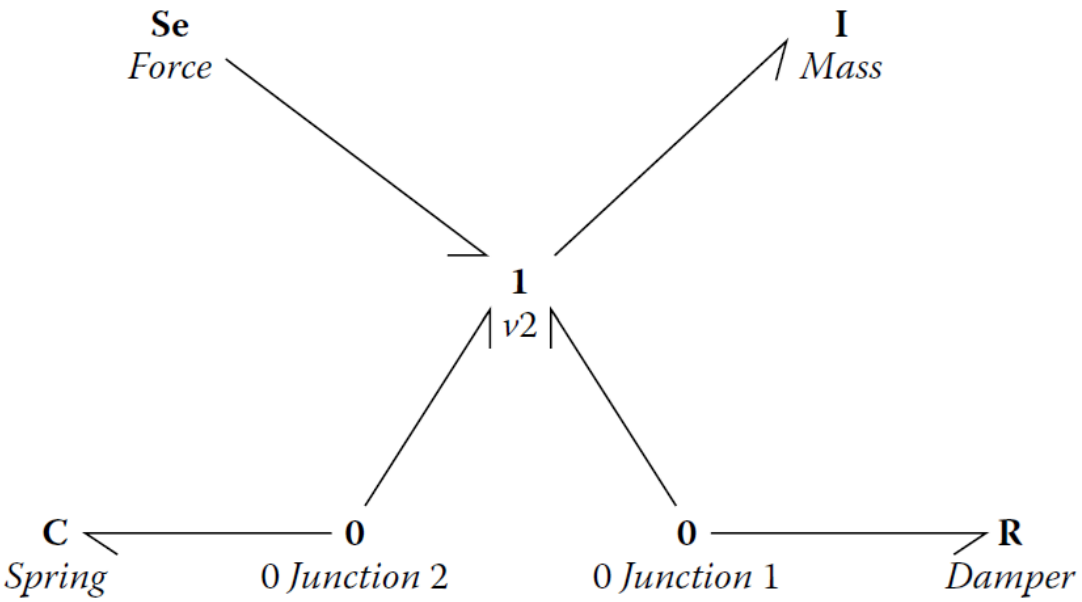


Figure 3.17: Simplified model

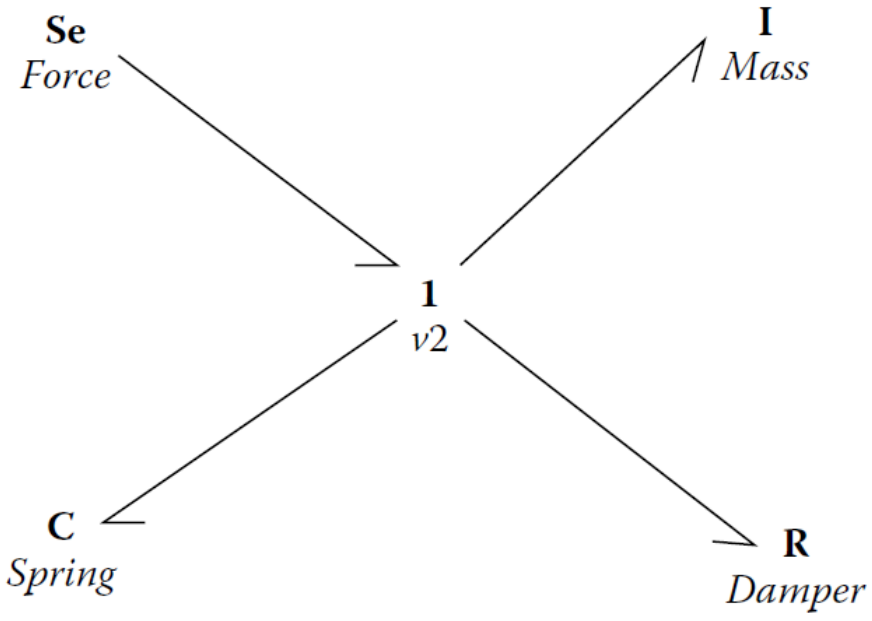


Figure 3.18: Final form of the bond graph representation.

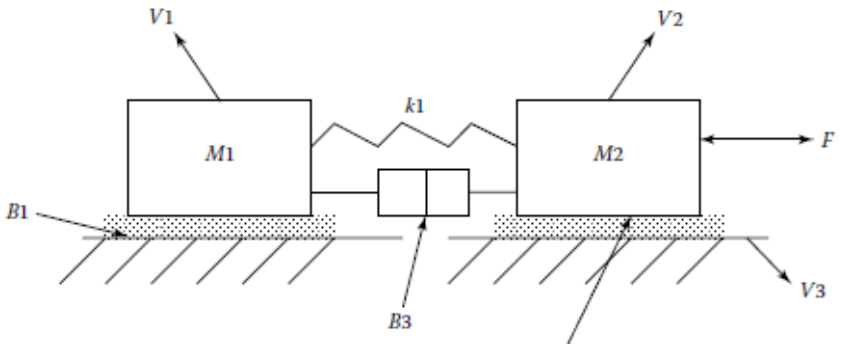


Figure 3.19: Mechanical model

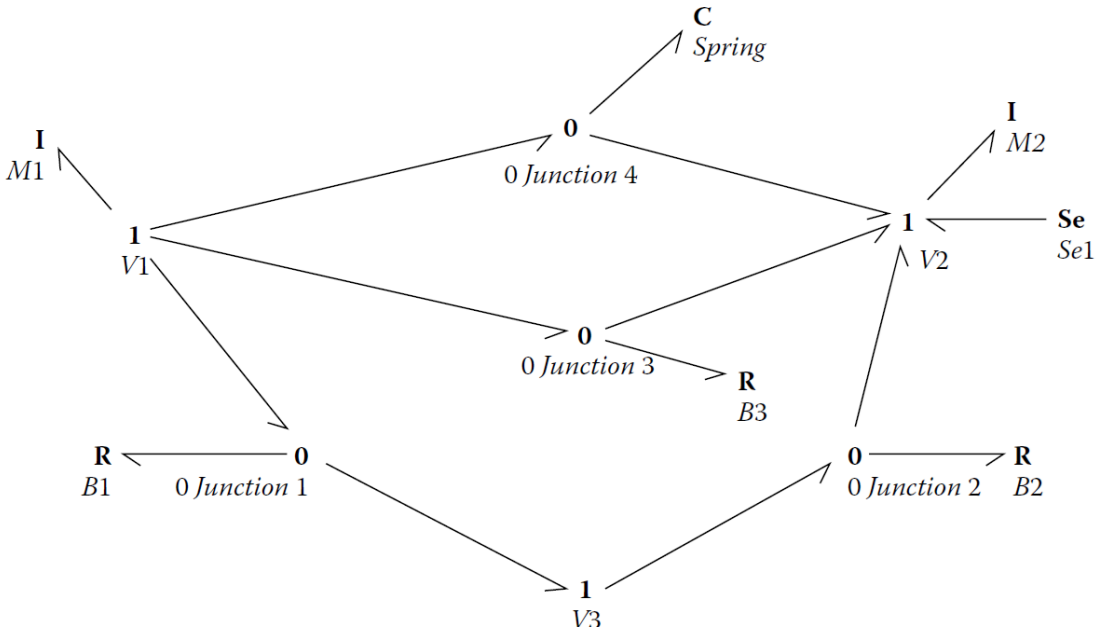


Figure 3.20: Initial bond graph representation

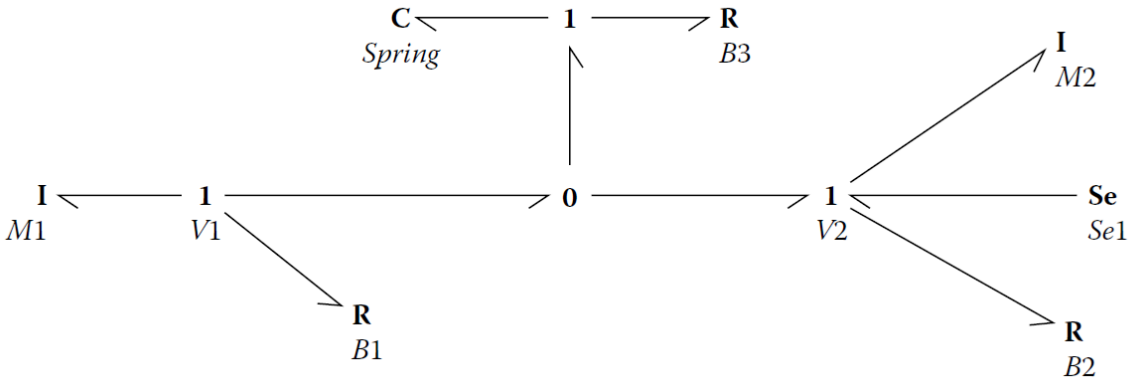


Figure 3.21: Final simplified bond graph

Example 03: Figure 3.22 depicts a mechanical system for which a bond graph representation is required. The system has several distinct velocity points, labeled

V_1, V_2, V_3, V_4, and V_5. The cable's flexibility around the pulley is modeled as a lumped spring K_1. The lever is treated as a massless transformer element, and the pulley is considered massless and frictionless. Since the spring at the other end and the damper are attached to different points, two transformers are needed, as illustrated in Figure 3.23. After simplification, the resulting bond graph is shown in Figure 3.24.

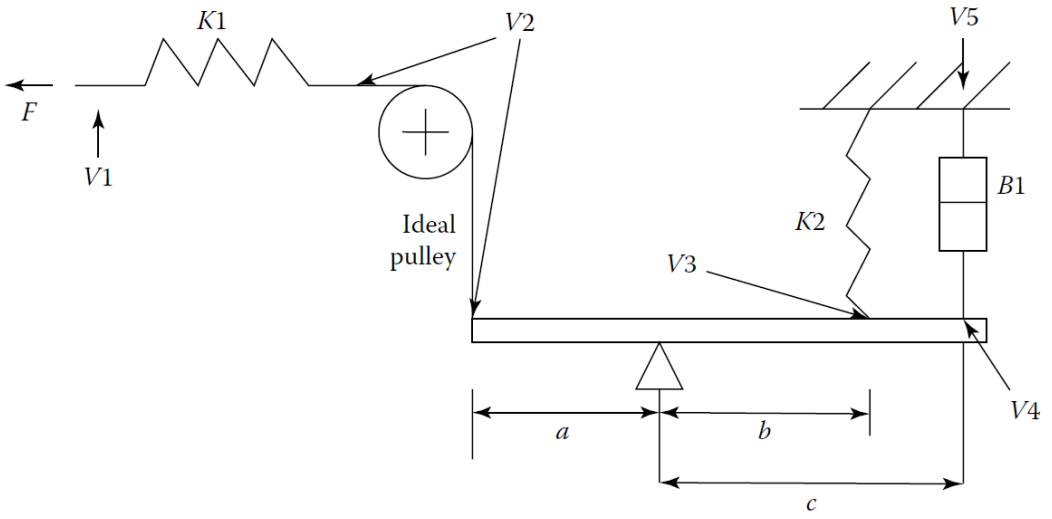


Figure 3.22: Schematic for a mechanical system.

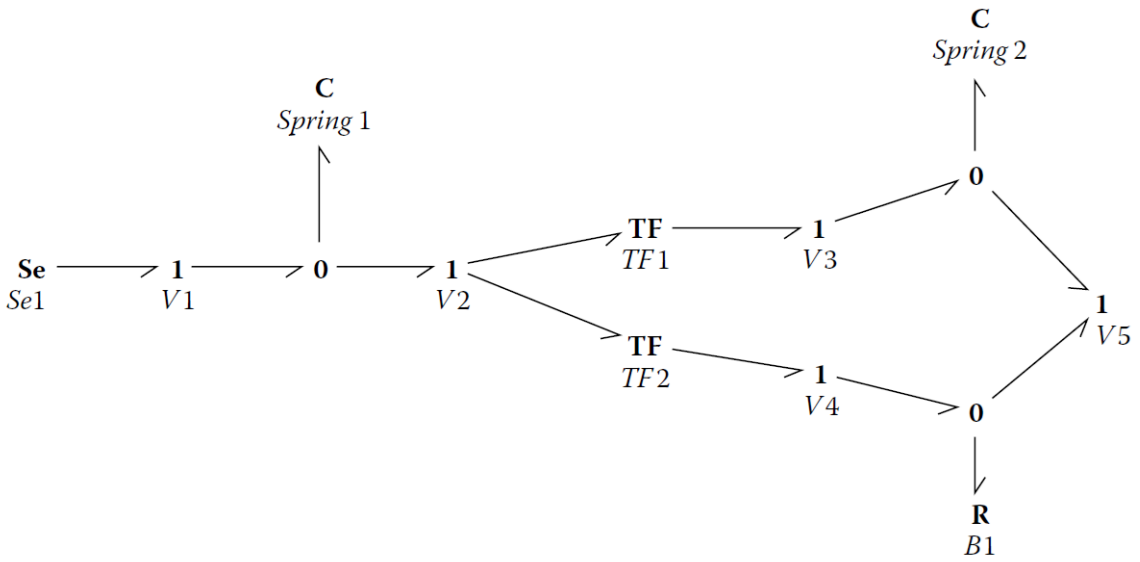


Figure 3.23: Initial bond graph model

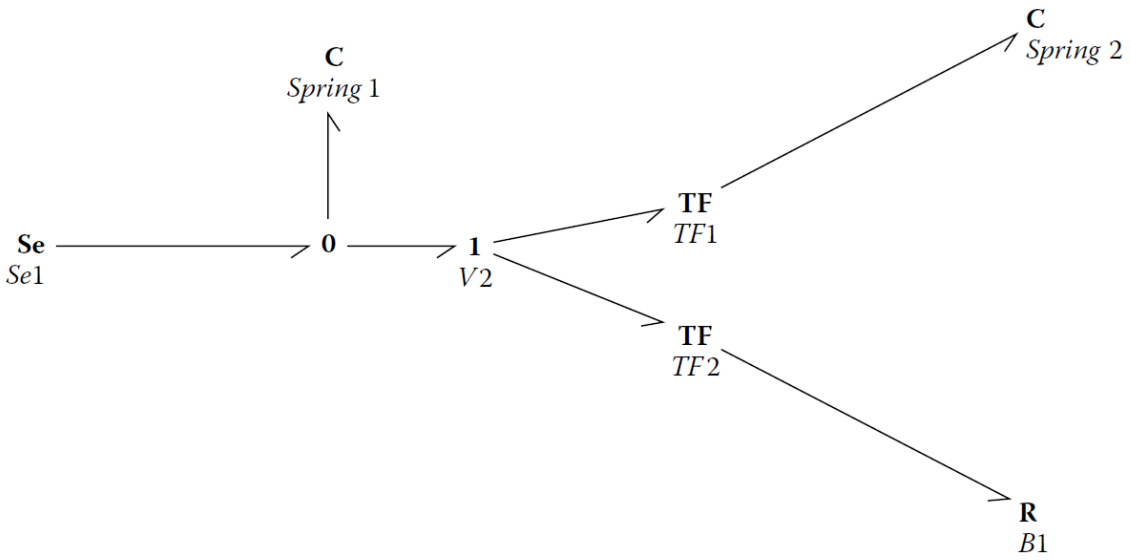


Figure 3.24: Final simplified model

3.5 Bond Graphs for Hydraulic Systems

The following steps can be used to construct the bond graph of a hydraulic system:

1. Identify the key components of the hydraulic system (reservoirs, pipes, pumps, valves, etc.).
2. Determine the flow variables (Q) and effort variables (P) for each component.
3. Represent each component using the appropriate bond graph element (R, C, I, Se, Sf).
4. Connect elements sharing the same pressure using 0-junctions.
5. Connect elements sharing the same volumetric flow using 1-junctions.
6. Include any transformers (TF) or gyrators (GY) if there are power conversions between different domains.
7. Simplify the bond graph by removing redundant junctions and applying causality rules.

Example 01: This example illustrates a simple hydraulic system in which fluid flows through a pipe into a tank. The fluid input is modeled as a constant effort source. The system accounts for the fluid's inertia in the pipe, viscous losses, and the tank's capacitance. Points a, b, and c indicate locations with different pressures.

To construct the bond graph, a 0-junction is assigned to each of these three points. The other elements are connected via 1-junctions, which link two adjacent 0-junctions according to the bond graph construction algorithm.

The initial bond graph representation is shown in Figure 3.26. A simplified version, obtained by removing the 0-junction representing atmospheric pressure (point c), is shown in Figure 3.27. The I (inertia) and R (resistive) elements between

points a and b are connected through the same 1-junction, as the flow rate in this section of the pipe is identical. The fluid inertia experiences resistance from the pipe walls.

$$C = \frac{A_t}{\rho g}$$

$$R = \frac{8\mu L}{\pi a^4}$$

$$I = \frac{\rho L}{A_p}$$

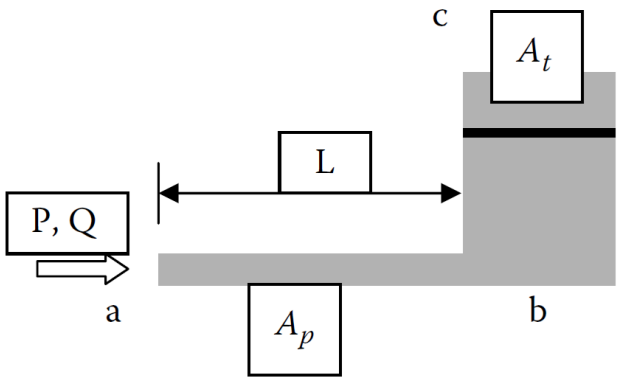


Figure 3.25: Hydraulic system

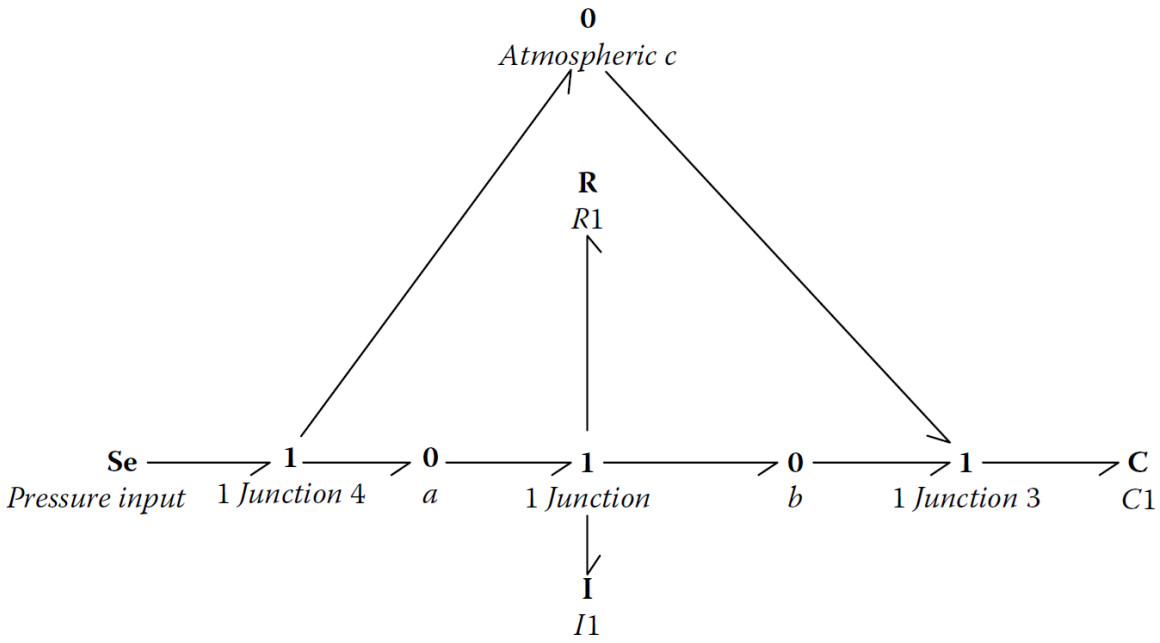


Figure 3.26: Initial bond graph

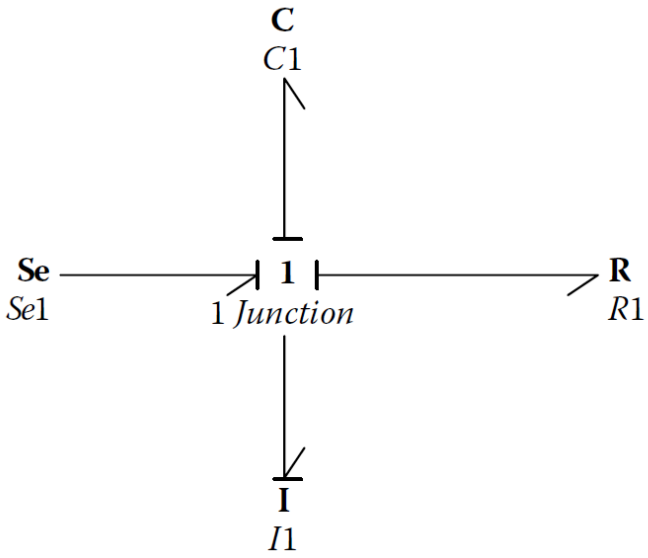


Figure 3.27: Final simplified bond graph

Example 02 :
This example illustrates fluid flowing through a pipe to fill two separate tanks, as shown in Figure 3.28. The pressures and flow rates at the inlet and outlet are indicated in the diagram. Following the previously described rules, the bond graph is first constructed to produce the initial model shown in Figure 3.29, which is then simplified to obtain the final representation depicted in Figure 3.30. The *I* elements model the fluid inertia in the three pipe sections, which can often be neglected, for instance, when the pipes are short. The *R* elements represent the resistances of the three pipe sections, and the *C* elements capture the energy stored in the tanks. The effort source (*Se*) corresponds to the pressure at the inlet.

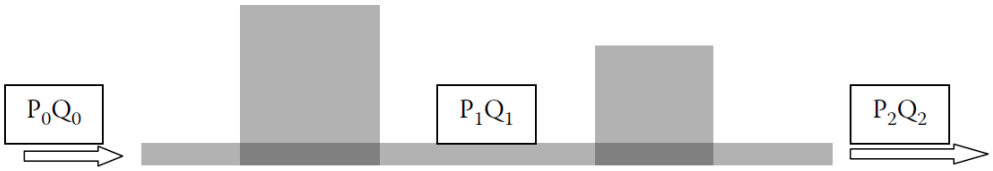


Figure 3.28: Hydraulic example

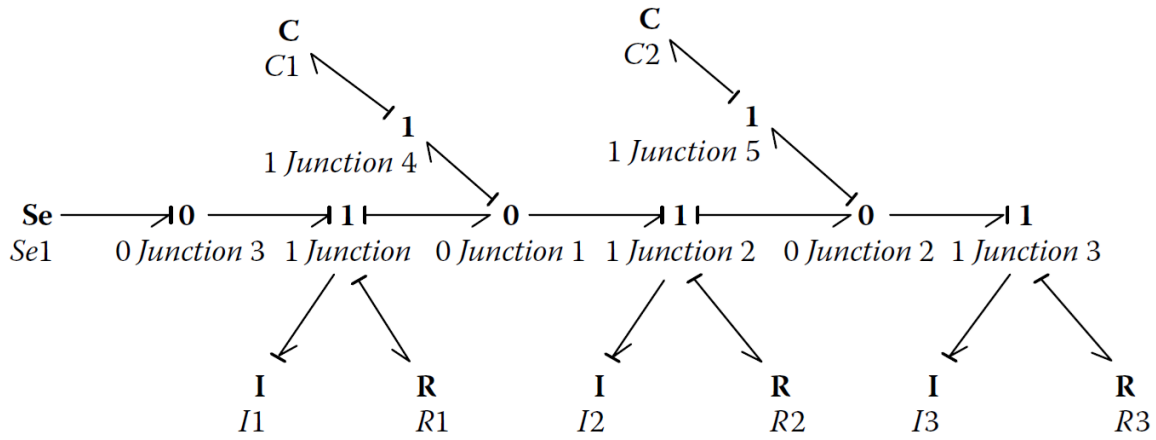


Figure 3.29: Initial bond graph

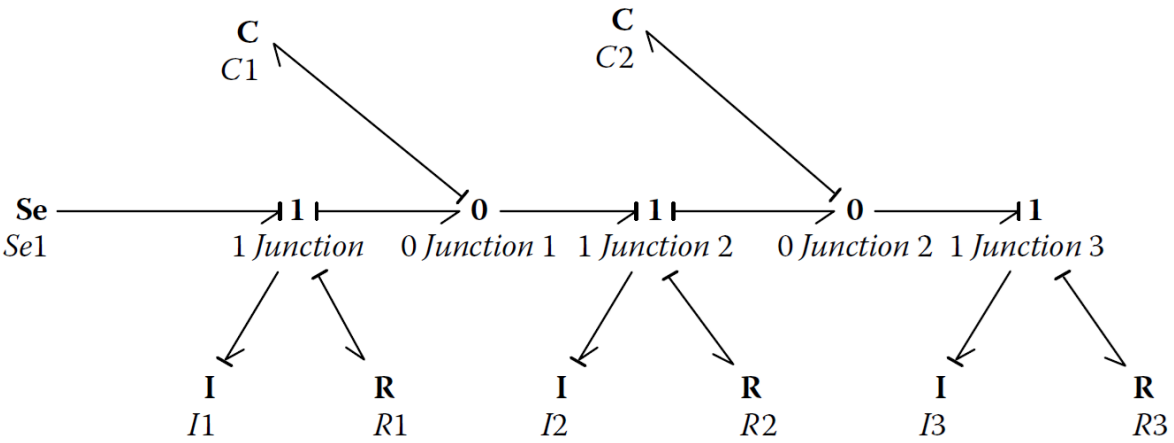


Figure 3.30: Final simplified bond graph

3.6 Causality

The term *causality* refers to the assignment of cause and effect in a system. Establishing causality is a crucial step for deriving the mathematical model of a system from its bond graph representation. To fully understand how causality operates within a bond graph, it is essential to recall several key concepts introduced earlier. These include:

1. The half-arrow indicates the direction of power flow.
2. Positive power flows in the direction of the half-arrow.
3. Although power is the product of effort and flow, the half-arrow does not represent the direction of either variable individually, but rather the direction of their product.

To develop the model equations, we need to understand more about the flow and effort variables. By convention, the flow and effort associated with each bond are denoted by the symbols f and e, placed above and below the bond, as shown in Figure 3.42. Causality is concerned with determining the direction of these flow and effort variables.

For example, consider a force applied to a mass. A source of effort is connected to the mass, imposing the force (effort), while the resulting velocity (flow) is determined by the mass's response. The effort information travels from the source to the mass, and the flow information returns from the mass to the source, as illustrated in Figure 3.31. The effort moves left to right, and the flow moves right to left.

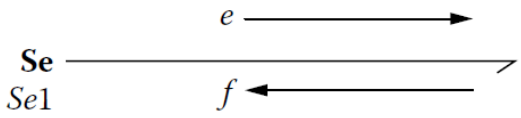


Figure 3.31: Effort and flow direction for an effort source.

For a two-port bond, each port carries information about either effort or flow. If effort travels from left to right, flow travels from right to left, since no element can control both simultaneously. Similarly, for a flow source, the flow is imposed on the system, while the resulting effort is determined by the system and returns to the source, as shown in Figure 3.32.

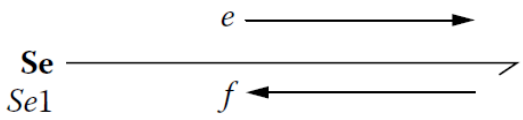


Figure 3.32: Effort and fl ow direction for a flow source.

In bond graphs, causality is indicated by a vertical line called the causal stroke, placed at the end of a bond toward which the effort flows. Since flow always travels in the opposite direction, its path need not be shown separately. Figure 3.33 illustrates the causal strokes for effort and flow sources.

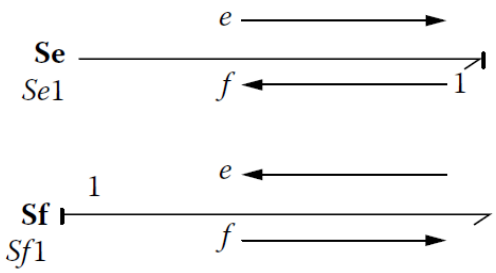


Figure 3.33: Enter Caption

3.6.1 Transformer

A transformer transmits power without storing or dissipating it. The flow and effort on one side are proportional to those on the other. Figure 3.34 shows the two possible causal structures: in the first, flow enters from the left; in the second, it enters from the right, with effort directions reversed.

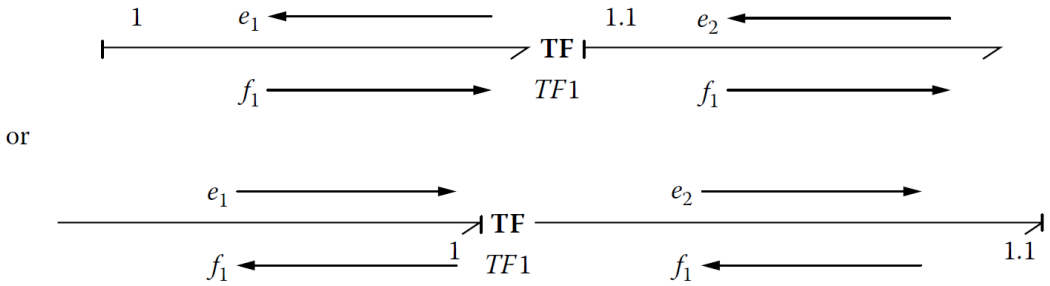


Figure 3.34: Causal structure of transformer element.

3.6.2 Gyrator

Like a transformer, a gyrator transmits power without storing or dissipating it. In a gyrator, the effort on one side determines the flow on the other, and vice versa. The two possible causal structures corresponding to the two opposite directions of effort and flow exchange.

3.6.3 Jonction

A 0-junction equalizes efforts across all connected bonds, while flows differ. One bond (the strong bond) provides the effort information, with its causal stroke near the junction; all others have strokes away from it, as shown in Figure 3.35.

A 1-junction equalizes flows across all bonds, while efforts differ. One strong bond provides the flow information, with its causal stroke away from the junction; all others have strokes near it, as shown in Figure 3.36.

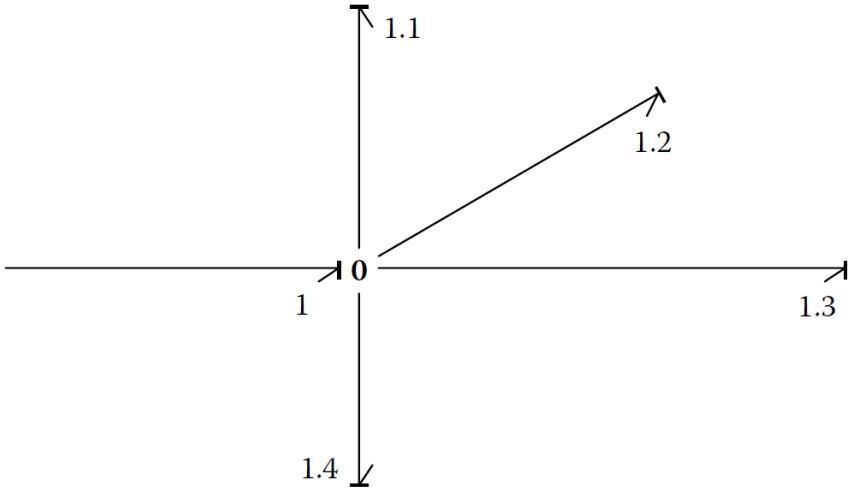


Figure 3.35: Causal structure for a 0 junction.

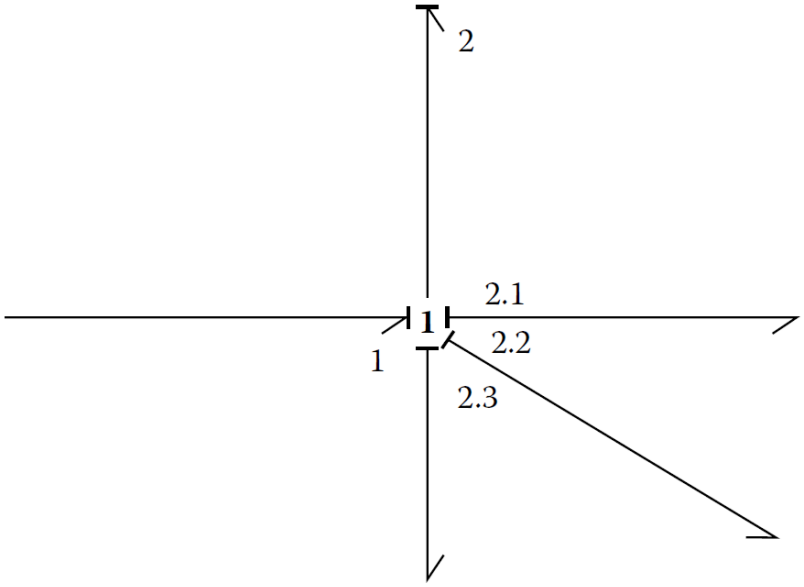


Figure 3.36: ausal structure for a 1 junction.

3.6.4 Storage Elements: I, C

Storage elements store and return energy without dissipation. Their causality can be either integral or differential. Integral causality, where effects result from the integration of past causes, is preferred. Differential causality, which depends on future causes, is less desirable but sometimes unavoidable. These concepts apply to both types of storage elements.

I, for Mass Elements or Inductances

The constitutive equation for the **I-element** may be written as:

$$\text{Momentum} = \text{Inertia} \times \text{Velocity} \quad (p = m \cdot v,\ \lambda = L \cdot i,\ \text{etc.})$$

In terms of generalized variables, this can be expressed as:

$$p = m * v \tag{3.19}$$

Therefore, after minor manipulation:

$$f = \frac{1}{m} \int e.dt \tag{3.20}$$

This is the meaning of integral causality for the I element.

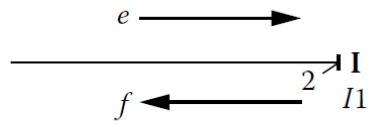


Figure 3.37: Integral causality for I element

The same constitutive equation for the I element may be rewritten in the following form:

$$e = m\frac{df}{dt} \tag{3.21}$$

This is the derivative form of the constitutive equation for the I element, and this representation is called the differential causality.

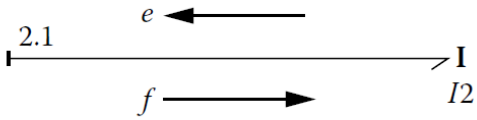


Figure 3.38: Differential causality for I element.

C, for Capacitive or Spring Elements

The constitutive equation for the **C-element** may be written as:

$$\text{Effort} = \text{Capacitive Constant} \times \text{Displacement} \quad (F = k \cdot x,\ V = \tfrac{1}{C} \cdot Q,\ \text{etc.})$$

In terms of generalized variables, this is written as:

$$e = \frac{q}{c} \tag{3.22}$$

Therefore,

$$e = \frac{1}{c} \int f dt \tag{3.23}$$

This is integral causality for the C element.

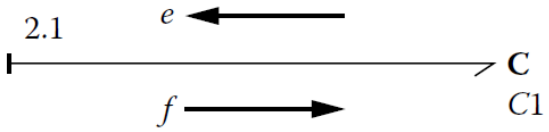


Figure 3.39: Integral causality for the C element.

The same constitutive equation for the C element may be written in the following form:

$$f = C\frac{de}{dt} \tag{3.24}$$

This is the derivative form of the constitutive equation for the C element, and this representa-tion is called the differential causality.

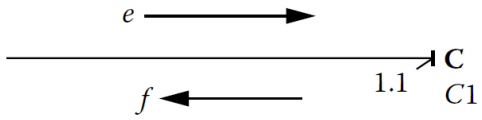


Figure 3.40: Differential causality for the C element.

3.6.5 R, for Resistive Elements

The resistance element does not store energy; it dissipates it. Its effort–flow relationship is purely algebraic, so it has neither integral nor differential forms. The causal stroke for an *R* element can appear on either side, as shown in Figure 3.41, meaning that effort or flow can be determined from the other depending on the rest of the system.

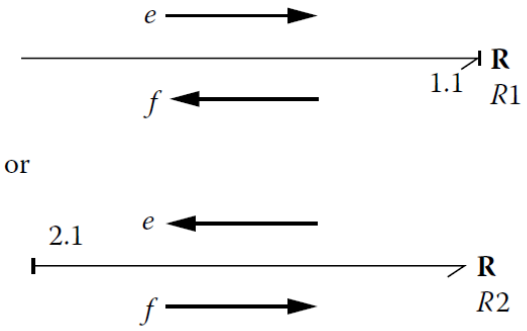


Figure 3.41: Causal structure for an R element.

3.6.6 Algorithm for Assigning Causality in a Bond Graph Model

1. Choose any source and assign its required causal stroke. Extend the causal implications through the bond graphs as far as possible using 0, 1, TF, GY, and

so forth.

2. Repeat this for all sources.
3. Choose the storage element (I or C) and assign its preferred causal-ity (integral causality). As in (1), extend the implications through the rest of the bond graphs.
4. Repeat step (3) for all storage elements.
5. Choose any unassigned R element, assign a causality to it (arbi-trary), and extend the implications throughout the bond graph.(Many R elements would be assigned causal strokes by this time.)
6. Repeat (5) for all unassigned Rs.
7. Choose any unassigned bond and assign arbitrarily and extend through the rest of the bond graph.
8. Repeat step (7) until you are done.

3.7 Conclusion

This chapter introduced the bond graph approach for electrical, mechanical, and hydraulic systems. Using effort and flow variables, bond graphs provide a unified and intuitive way to represent energy exchange and system dynamics. The concept of causality defines how effort and flow interact, guiding the correct formulation of system equations. Overall, bond graphs offer a consistent method for modeling and analyzing multi-domain dynamic systems.

CHAPTER 4

Structural Analysis Based on Bond Graph Modeling

4.1 Introduction

Structural analysis based on bond graph modeling provides a systematic way to understand the internal organization and energy interactions of complex physical systems. By representing the flow of power through interconnected elements, bond graphs reveal the structural dependencies, causality, and dynamic behavior of multidisciplinary systems in a unified framework.

4.2 Rank-hg of the State Matrix Associated with the Bond Graph

The order n of a model — and thus the dimension of the state matrix — is equal to the number of I and C elements that exhibit **integral causality** when the bond graph is assigned **preferential integral causality**.

The rank q of the state matrix corresponds to the number of **non-zero modes** of the model, and it can be expressed as:

$$q = n - k \tag{4.1}$$

where k is the number of zero modes. This can also be expressed by writing the transfer function in the form:

$$H(p) = \frac{N(p)}{D(p)} = \frac{N(p)}{p^k D_1(p)} \tag{4.2}$$

with $\deg(D(p)) = n$ and $\deg(D_1(p)) = q$. Knowing the rank of matrix **A** is of immediate interest for control system design (steady-state behavior, control by inversion, etc.).

Theorem.
The rank-hg of the state matrix associated with a bond graph model initially in integral causality is equal to the number of I and C elements in integral causality that admit a derivative causality when derivative causality is applied to the bond graph. If (n_{Ji}) (respectively (n_{Jd})) represents the number of I and C elements in integral causality when an integral causality i (respectively derivative d) is applied to the bond graph.

Remark.
It should be noted that the bond-graph rank (rank_{bg}) is actually the true rank, as it takes into account the dependency relationships between the terms of the matrix, unlike conventional structural methods.

Example : As an example, let us consider the following bond graph:

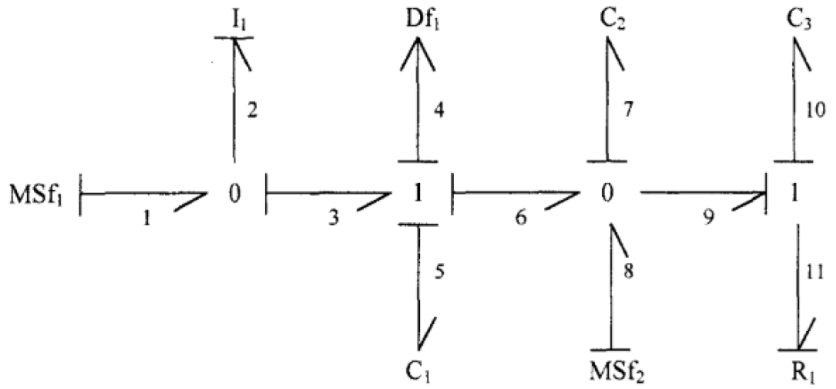


Figure 4.1: The bond graph represented in integral causality corresponds to a system of order four.

The four I and C elements admit an integral causality without creating any conflict (Figure 4.1). Therefore, the model is of order $n = (n_i)_i = 4$.

We now assign derivative causality to the same bond graph model:

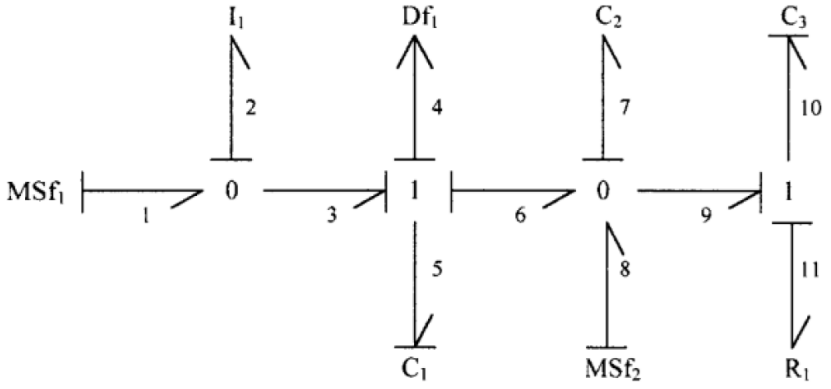


Figure 4.2: The bond graph represented in derivative causality.

If we try to assign derivative causality to all the I and C elements, we find that it is not possible to switch the element C_2 to derivative causality (we could also have chosen C_1 or C_3). We can therefore deduce that $(n_i)_d = 1$, and that the bond-graph rank of the state matrix is $q = 3$.

4.3 Structural Controllability and Observability

The following theorem allows us to determine graphically whether a bond graph model is structurally controllable or observable. It relies on the concept of *dualization* of sources and sensors, which consists in transforming a source (or a detector) of effort (respectively of flow) into a source (or a detector) of flow (respectively of effort). A bond graph model is said to be **structurally controllable** (respectively **observable**) if and only if:

1. There exists a causal path linking a control source (respectively a sensor) to every dynamic element I and C in integral causality when the bond graph is set in integral causality mode.

2. All I and C elements admit a derivative causality when the bond graph is placed in preferential derivative causality mode. If some I or C elements remain in integral causality, the *dualization* of the sources (respectively the sensors) must make it possible to switch them to derivative causality.

Example. Consider the bond graph of a simple mechanical system composed of a mass–spring–damper, as shown in Figure **??**. The system is actuated by a force source Se and the velocity of the mass is measured.

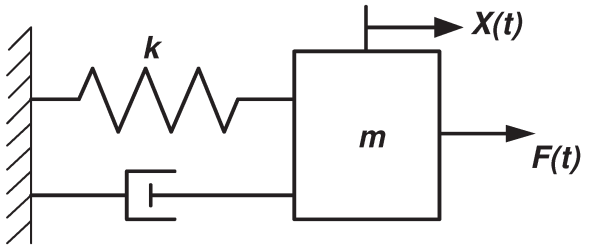


Figure 4.3: Mass–Spring–Damper System

- In **integral causality**, the mass (I) and spring (C) elements both receive integral causality without conflict. A causal path exists from the effort source Se (input) to both I and C elements, ensuring **structural controllability**.

- In **derivative causality**, both I and C elements can switch to derivative causality without conflict. This means that the velocity sensor (output) ensures **structural observability**.

Hence, according to the structural conditions, the mass–spring–damper system is both **structurally controllable and observable**.

Example: Hydraulic Actuator with Mechanical Load

Consider a hydraulic actuator driving a mechanical load. The bond graph model includes:

- A pressure source Se representing the hydraulic pump,

- A resistance R_h modeling valve losses,
- A compliance C_h for the fluid compressibility,
- An inertia I_m representing the mechanical load,
- A mechanical resistance R_m modeling friction,
- A sensor *De* measuring piston displacement.

The bond graph can be represented schematically as:

$$Se \; -- \; R_h \; -- \; C_h \; -- \; I_m \; -- \; R_m \; -- \; 0\text{-junction} \; -- \; De$$

Integral Causality: Both C_h and I_m admit integral causality, so the system order is $n = 2$. The state variables are the fluid pressure q_h and the load momentum p_m.

Derivative Causality: Under preferential derivative causality, both C_h and I_m can switch to derivative form, indicating that the system is fully controllable with a single input source *Se*.

Observability: The displacement sensor *De* provides direct information about the load position. Dualization confirms that the system is also fully observable.

Remark.

- When all the *I* and *C* elements initially in integral causality can admit a derivative causality, a single control input is sufficient to make the model controllable, and its placement in the model depends only on technological considerations.
- When k elements I, C remain in integral causality, then k properly placed control sources are required to make the model controllable. The dualization of these sources allows verification of whether the chosen placement is correct.
- The same reasoning applies identically for the study of observability.

Degree of Observability

The *degree of observability*, denoted q_{obs}, represents the number of independent state variables that can be reconstructed from the system outputs. In the bond graph approach, it corresponds to the number of dynamic elements (*I* and *C*) whose states are observable through causal paths leading to sensors.

Graphical Analysis:

1. Assign integral causality to all dynamic elements (I, C);
2. Identify all sensors (De, Df);
3. Each dynamic element is observable if a causal path exists from it to a sensor;

Thus, the degree of observability is given by:

$$q_{obs} \leq n$$

where n is the system order. If all states are observable, then:

$$q_{obs} = n$$

and the system is said to be *fully observable*.

4.3.1 Classification of System Variables

Classification of System Variables

In a bond graph model, two types of detectors are used: the *effort detector* and the *flow detector*, which are employed to measure the system's elements (variables). We can distinguish two main types of variables (elements): the **measured variables** and the **deducible variables**.

- **Measured variables:** These are connected to the sensor through a direct causal path, allowing their values to be read directly using the detectors.
- **Deducible variables:** These are variables whose values can only be determined through other variables. They are connected to the detectors by indirect causal paths.

Rule 1. A measured variable is considered *estimable* (redundant) if it belongs to at least two direct causal paths leading to two different sensors.

Rule 2. A measured variable is considered *non-estimable* if it belongs to a single causal path leading to only one sensor.

Rule 3. A variable is considered *unmeasured and deductible* if it belongs to at least two indirect causal paths leading to a sensor.

Rule 4. A variable is considered *non-measurable and non-deductible* (non-estimable) if there is no path leading to any sensor.

Principle. Variables that are part of a causal path leading to a detector have zero redundancy (they have a zero degree of observability).

4.4 Definition of the Degree of Observability (Degree of Redundancy)

The concept of the degree of observability is essential for distinguishing observable variables from those that are not. For a given variable, this degree corresponds to the number of detectors connected to that variable, minus one. Thus, a variable connected to at most one detector is considered observable. Moreover, the degree of redundancy represents the number of detectors (or sensors) linked to the variable through disjoint causal paths, allowing an evaluation of the robustness of the observation.

We distinguish two levels of redundancy, namely:

4.4.1 Minimal observability

A variable is redundant of degree 0 if there exists at least one (or more) configuration(s) in which a faulty detector in the process makes the concerned variable inaccessible.

4.4.2 Observability of arbitrary degree

By extension of the previous concept, a variable is redundant of degree k (or observable of degree k) if it is an observable variable whose value remains deducible even during the simultaneous failure of any k sensors in the system.

Lemma. If a variable has an observability degree of zero, then it is observable. If it has an observability degree equal to $k \geq 1$, then it is monitorable. These notions apply to any process containing numerous variables with varying degrees of observability. Indeed, the observability degree of a system is defined as the minimum of the degrees of its components. Thus, it can be concluded that an observable process has an observability degree of zero, and a process with a degree $k > 0$ is monitorable. The above definitions serve to define a variable according to its degree of redundancy, which reflects its availability in relation to the placement of sensors.

4.4.3 Monitorability

. The installation of a sensor is essential for designing integrated diagnostic systems, due to the fact that the monitorability of an industrial process is linked to the number of sensors and their positioning (placement). The nature of sensor placement is either combinatorial or graphical (manual) on the bond graph.

Example Consider the hydraulic system consisting of 2 tanks shown in Figure 4.4. The bond graph model associated with this system is represented in Figure 4.5.

In this section, we will suggest the implementation of sensors to improve the structural characteristics of the system as well as its monitoring capability, despite a zero observability degree. In this two-tank system, where each component is monitored by a single detector, the procedure is illustrated in the figure 4.6

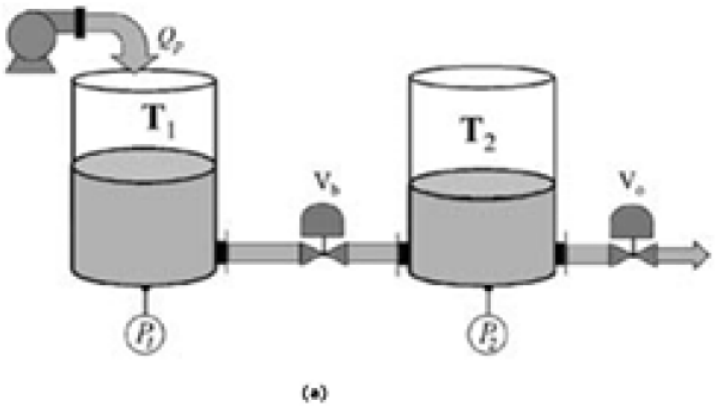


Figure 4.4: Hydraulic system

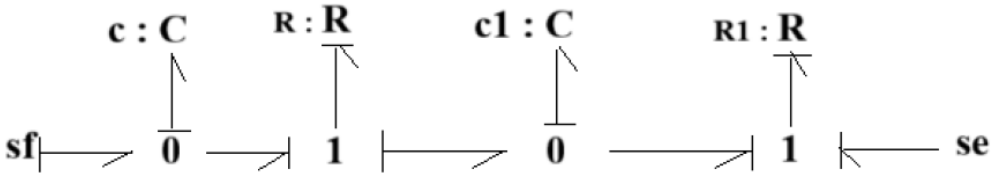


Figure 4.5: Bond graph model

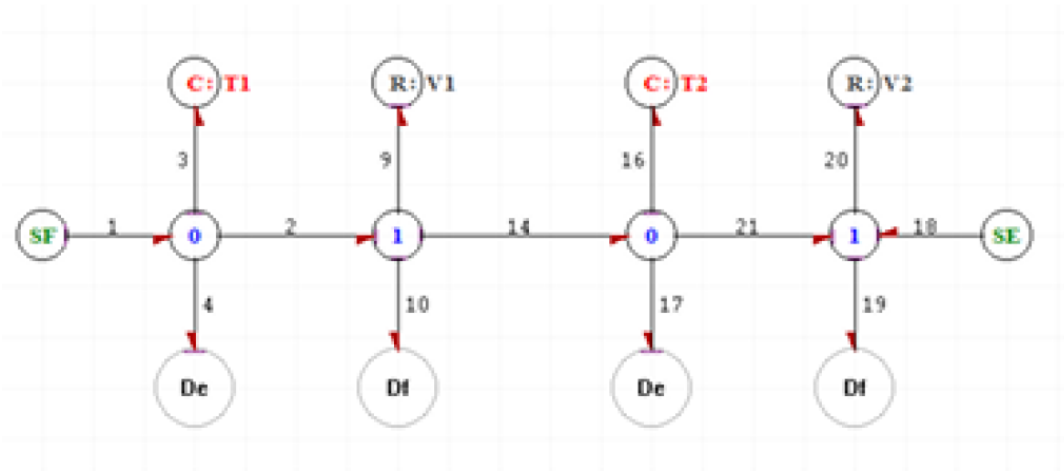


Figure 4.6: Bond Graph Model with Detectors

It is assumed that systems modeled by bond graphs are structurally controllable if the following conditions are met:

1. The establishment of a causal path connecting a source (or a sensor) to each unit. Dynamic elements I and C when the bond graph is assigned integral causality.

2. The elements I and C have derived causality when the bond graph is differentiated, and when duality is applied to the sources.

Regarding structural observability, the term **source** is replaced by **sensor** in the above conditions.

Observability Degree

The element $C : T_1$ (or $C : T_2$) has degree zero (0 degree of redundancy) since it is associated with only one detector. This variable is measured, not estimable, because it follows a unique causal path leading to a single detector.

We can observe the variable $C : T_1$ because its observability degree is zero.

The element $R : V_1$ (or $R : V_2$) has degree zero (0 degree of redundancy) since it is associated with a single detector; this variable is measured and not estimable because it belongs to only one causal path leading to a single detector.

The variable $R : V_1$ is observable since its observability degree is zero.

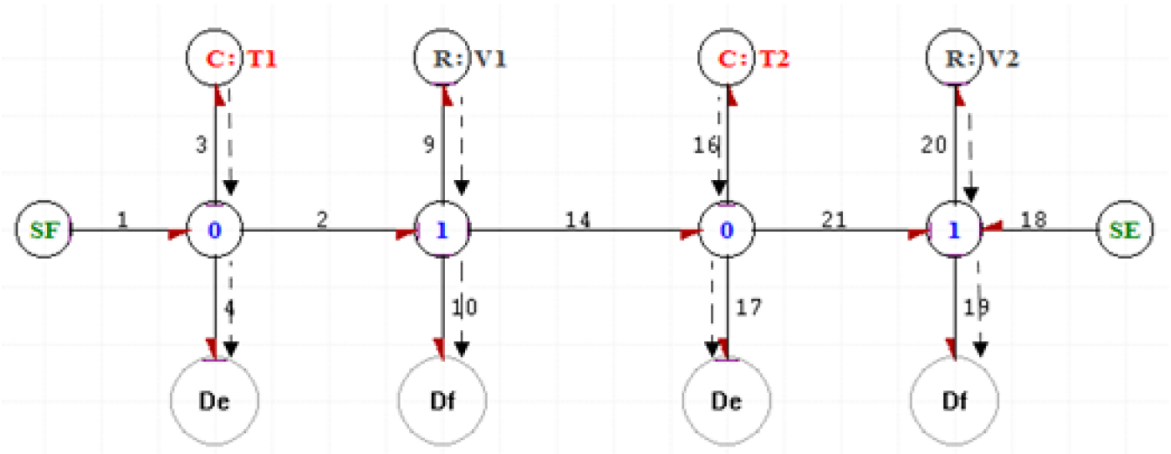


Figure 4.7: Causal paths of the sensors

It is therefore possible to state that all the components of our system are observable, given that their observability degrees are zero.
To ensure monitorability, it is necessary to improve the observability of the variables. Making variables observable — Observability is an essential but not sufficient condition for monitorability.

Diagnosis using the qualitative bond graph approach After the installation of sensors, the structural analysis of the two-tank hydraulic system provides the following results:

- It indicates that the system has rank 2 and also order 2.
- It indicates that there is no control point, and that the system is not controllable, but it can be easily made controllable by making our sources configurable (that is, instead of Sf or Se, we convert them into MSe, and possibly into MSf).
- It ensures that the system is structurally observable and that its states have a causal path to one or more detectors.

Structural Analysis Results Mode Test

- System order: 2

- Bond Graph rank of the system: 2
- There are no structurally null nodes in the system
- It is sufficient to observe this system with a properly placed observer only if the system is not disjoint

Controllability Test

- There are no (modulated) control sources in this model
- The system is structurally (and therefore always) uncontrollable without control sources

Observability Test

- Number of observers in the system: 4
- All states have a causal path to at least one observer
- The system is structurally observable
- The following observers are superfluous:
    1. Observer at bond number 10
    2. Observer at bond number 19

4.5 Conclusion

In this chapter, we presented a structural analysis based on bond graph modeling. The rank of the state matrix associated with the bond graph was examined to assess the system's structural controllability and observability. The degree of observability was evaluated, highlighting the system's ability to be monitored and diagnosed effectively. This approach provides a rigorous framework for understanding the system's structural properties and lays the groundwork for improving its controllability and monitorability through sensor placement and system design.

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