Nonlinear Control system

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May 20, 2025

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0.1 Preface

Automatic control systems are pervasive across many industrial sectors, including aerospace, aeronautics, automotive engineering, and process engineering. Developing tools for analyzing and designing control and observation laws is essential to ensure the proper functioning of these systems.

Initially, such tools were designed for linear or linearized systems. However, approximating a system by its tangent linearization often proves insufficient—or even unsuitable—for most real-world systems, which exhibit inherently nonlinear behavior on a global scale. Therefore, it is crucial to consider, study, and develop dedicated tools for nonlinear systems.

The aim of this course is to highlight the fundamental differences between linear and nonlinear systems, to present methods for analyzing nonlinear systems, and to introduce selected techniques for synthesizing control laws for such systems.

To facilitate understanding, this course material introduces a collection of classical results, often scattered across various references. The purpose is not to provide an exhaustive review of all research in the field, but rather to emphasize the key contributions that form the foundational basis of nonlinear systems theory. These results are particularly useful as a starting point for master's and postgraduate students seeking to explore this subject. Notably, the course draws inspiration from well-established works such as "Nonlinear Systems" by H. Khalil and "Nonlinear Systems Analysis" by J.-J. Slotine.

This approach, which I have successfully applied for several years, offers a good balance: it saves time while encouraging student engagement by having them complete the missing parts during the course. This also fosters better interaction in class.

As with any work, there is always room for improvement. I welcome feedback, corrections, or suggestions from readers. Please feel free to reach out to the email address provided on the cover page. Thank you in advance for your contributions.

Chapter 1

Introduction to theory: basic definitions

1.1 System Description

Nonlinear systems can generally be described by a vector differential equation

$$\dot{x}(t) = f(x, u) \tag{1.1}$$

where

- $x = [x_1, \cdot, x_n]^T \in \mathbb{R}^n$ is the state vector, n representing the order of the system,
- $u = [u_1, \cdot, u_m] \in \mathbb{R}^m$ is the input vector, which may include control inputs and external disturbances,
- $f(.)$ is a vector field in $\mathbb{R}^n$, where the function associates a vector to an n-dimensional point x.

Equation (1.1) is known as the state equation. Its solution takes the form $x(t_0, t)$, which defines a family of time trajectories in the state space (also referred to as the phase space). By imposing the initial condition $x(t_0)$, one unique trajectory is determined.

Additionally, there is another equation called the output equation, which typically relates the system's state $x(t)$ to the observed output of the system, denoted as:

$$y = h(x, u) \tag{1.2}$$

where $y(t) \in \mathbb{R}^p$ is the output vector, and $h(.)$ is a function that maps the state and input vectors to the output.

Equations. (1.1) and (1.2) together are called the state-space model.

An affine control system is a mathematical representation of a dynamical system where the dynamics are affine in the control inputs. The general form of an affine control system is:

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ y &= h(x) \end{aligned}$$

where

- $f(x) \in \mathbb{R}^n$ is the drift term or uncontrolled dynamics, describing how the system evolves when no control input is applied (u = 0).
- $g_i(x) \in \mathbb{R}^n$ for $i = \{1, \ldots, m\}$ are vector fields that describe how each control input u_i influences the system's dynamics.

Example of an Affine Control System

Consider a simple example of a two-dimensional vehicle moving in a plane with the following dynamics:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= u_1,\\ \dot{x}_3 &= u_2,\end{aligned}$$

where:

- x_1 represents the position of the vehicle along one axis,
- x_2 represents the velocity along that axis,
- x_3 represents the position along another axis,
- u_1 and u_2 are the control inputs, representing acceleration along the respective axes.

In vector form, this system can be written as:

$$\dot{x}(t) = \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2(t),$$

where:

$$f(x) = \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix} \quad \text{is the drift term,}$$

$$g_1(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{are the control vector fields.}$$

The general form of an linear control system is:

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned} \tag{1.3}$$

We define $A \in \mathbb{R}^{n\times n}$ as the system matrix, $B \in \mathbb{R}^{n\times m}$ as the input matrix, $C \in \mathbb{R}^{r\times n}$ as the output matrix, and $D \in \mathbb{R}^{r\times m}$ as the direct transmission matrix.

1.1.1 Nonlinear System Examples

1. **Systems with essential nonlinearities in the model**

   Exemple : Self-balancing vehicle

$$\begin{aligned}(M+m)\ddot{x} &= ml\dot{\theta}^2\sin(\theta) - mg\sin(\theta)\cos(\theta) + u\\ (I+ml^2)\ddot{\theta} &= mgl\sin(\theta) - ml\cos(\theta)(\ddot{x} + l\dot{\theta}^2\sin(\theta))\end{aligned}$$

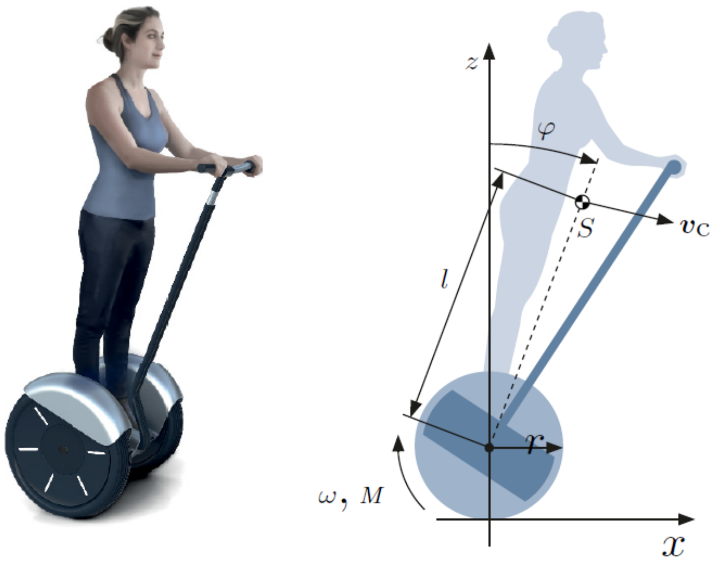


Figure 1.1: Self-balancing vehicle

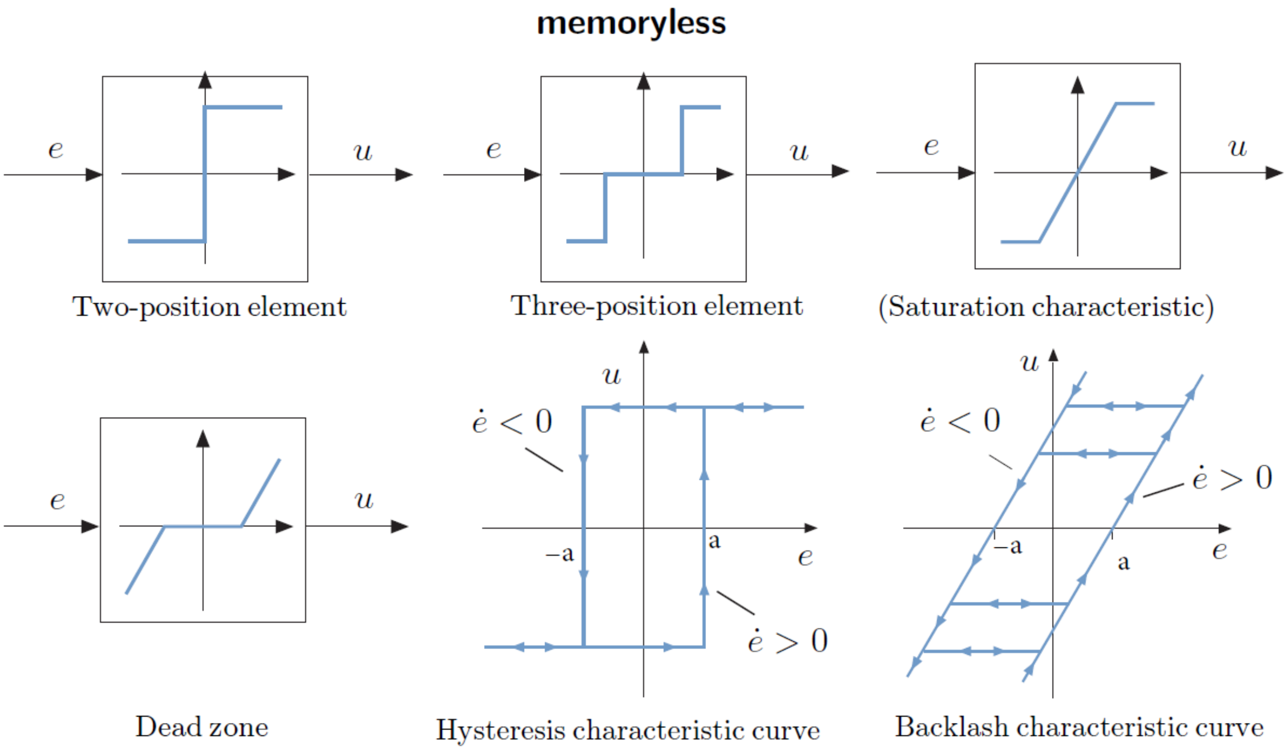
2. **Systems with saturation**

$$\begin{aligned} \dot{x} &= Ax + Bsat(u) \\ u &= PID(x) \end{aligned}$$

$$\boxed{sat(u) = \begin{cases} u & \text{If} \quad |u| \leq 1 \\ sgn(u) & \text{If} \quad |u| \geq 1 \end{cases}}$$

- The output is proportional to input for a limited range.
- Output becomes constant if input is outside this range.

3. **Common Nonlinearities**



Two-position element

Three-position element

(Saturation characteristic)

Dead zone

Hysteresis characteristic curve

Backlash characteristic curve

Figure 1.2: Common Nonlinearities

1.1.2 Superposition principals:

For linear systems

- The Linear Systems satisfy the superposition principals:
  1. Homogeneity: $f(\alpha x) = \alpha f(x)$, $\forall \alpha \in \mathbb{R}$.
  2. Additivity: $f(x + y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}^n$.
- Unique equilibrium point.
- Stability is independent of initial conditions $x(t) = x(0)e^{At}$

For nonlinear systems

Non superposition principle $\implies$ More complex behavior

Example : Under-water vehicle $\dot{v} + |v|v = u$

- Settles faster in response to positive step.

- Scaling input does not result into the same scaling in output.

!h

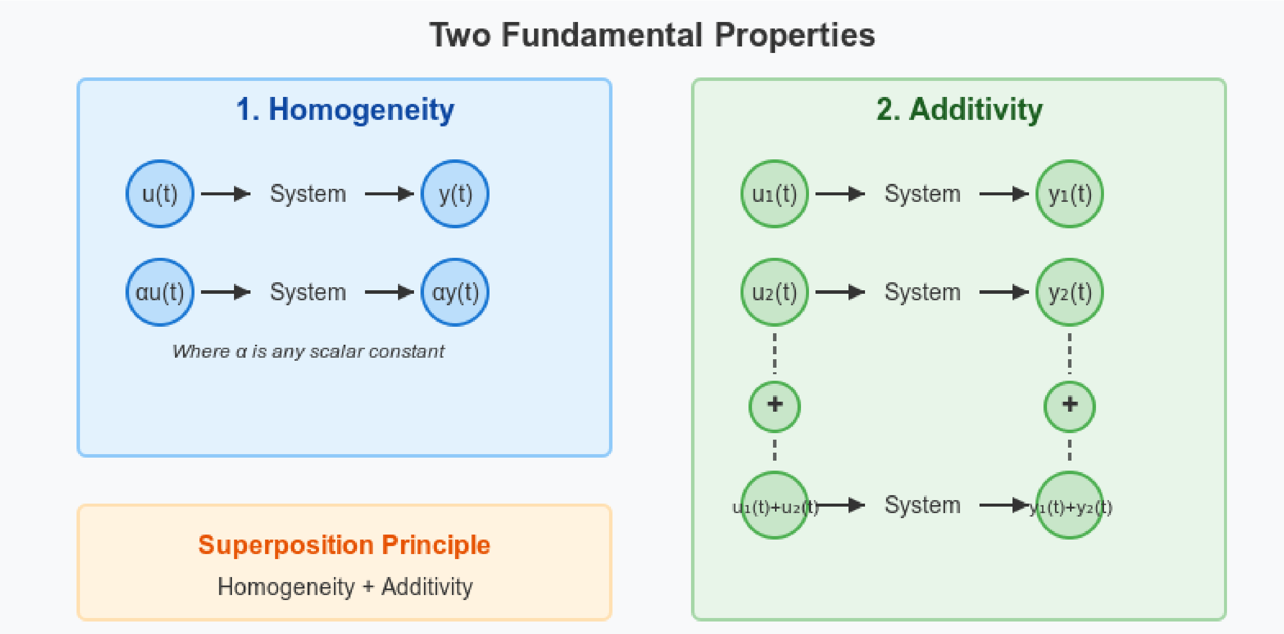


Figure 1.3: Superposition principals:

$$u = 1$$

$$\begin{aligned} \Rightarrow 0 + |v_s|v_s &= 1 \\ \Rightarrow v_s &= 1 \end{aligned}$$

$$u = 10$$

$$\begin{aligned} \Rightarrow 0 + |v_s|v_s &= 10 \\ \Rightarrow v_s &= \sqrt{10} \end{aligned}$$

1.1.3 Equilibrium Points of Nonlinear Systems:

An **equilibrium point** represents a stationary condition of a dynamical system. The state $x_e \in \mathbb{R}^n$ is a fixed point for $\dot{x} = f(x)$ if $f(x_e) = 0, \forall t \geq 0$.

This means that if the system starts at x_e, it will remain there for all future times (assuming no external perturbations).

Example: Finding Equilibrium Points of a Nonlinear System

Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2, \\ \dot{x}_2 &= -x_2 + x_1 x_2. \end{aligned}$$

To find the equilibrium points, set $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$:

$$\begin{aligned} -x_1 + x_2^2 &= 0 \quad (1) \\ -x_2 + x_1 x_2 &= 0 \quad (2) \end{aligned}$$

From Equation (2), factorize:

$$x_2(-1 + x_1) = 0.$$

This gives two cases:

- $x_2 = 0$,
- $x_1 = 1$.

Case 1: $x_2 = 0$
Substitute $x_2 = 0$ into Equation (1):

$$-x_1 + 0^2 = 0 \quad \Rightarrow \quad x_1 = 0.$$

Thus, one equilibrium point is:

$$x_e^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Case 2: $x_1 = 1$
Substitute $x_1 = 1$ into Equation (1):

$$-1 + x_2^2 = 0 \quad \Rightarrow \quad x_2^2 = 1 \quad \Rightarrow \quad x_2 = \pm 1.$$

Thus, two additional equilibrium points are:

$$x_e^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_e^{(3)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The equilibrium points of the system are:

$$x_e^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x_e^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_e^{(3)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

1.1.4 Equilibrium Points of linear Systems:

A general linear time-invariant (LTI) system is described by:

$$\dot{x} = Ax + Bu,$$

If there is no external input ($u = 0$), the system reduces to:

$$\dot{x}(t) = Ax(t).$$

An equilibrium point $x_e \in \mathbb{R}^n$ satisfies the condition:

$$\dot{x} = 0 \quad \Rightarrow \quad Ax_e = 0.$$

This means that at an equilibrium point, the state x_e lies in the null space of the matrix A.

The solutions to $Ax_e = 0$ form the null space (kernel) of A. Depending on the rank of A, the null space may contain multiple solutions.

- If A is invertible (full rank), the only solution is $x_e = 0$, meaning the origin is the unique equilibrium point.
- If A is singular (not full rank), there are infinitely many equilibrium points corresponding to the basis vectors of the null space of A.

Exemple Consider the following linear system:

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x(t).$$

To find the equilibrium points, solve $Ax_e = 0$. The equation $Ax_e = 0$ becomes:

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{e,1} \\ x_{e,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the system of equations:

$$\begin{array}{rcll} -x_{e,1} + x_{e,2} & = & 0 & \text{(e1)}, \\ 0 & = & 0 & \text{(e2)}. \end{array}$$

From Equation (e1):

$$x_{e,2} = x_{e,1}.$$

Thus, the equilibrium points satisfy:

$$x_e = \begin{bmatrix} x_{e,1} \\ x_{e,2} \end{bmatrix} = \begin{bmatrix} x_{e,1} \\ x_{e,1} \end{bmatrix} = x_{e,1} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $x_{e,1} \in \mathbb{R}$ is a free parameter.

The equilibrium points of the system are all scalar multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In other words, the equilibrium points form a line in the state space:

$$x_e = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c \in \mathbb{R}.$$

1.1.5 Linear approximations around an equilibrium

A linear approximation of a system around an equilibrium point can be used to study the behavior of a nonlinear system around the equilibrium point.

To approximate the nonlinear system near x_e, we perform a Taylor expansion of $f(x)$ around x_e. For small deviations $\delta x = x - x_e$, the Taylor expansion gives:

$$f(x) \approx f(x_e) + J(x_e)\delta x + \text{higher-order terms},$$

where $J(x_e)$ is the Jacobian matrix of $f(x)$ evaluated at x_e, i.e.,

$$J(x_e) = \left. \frac{\partial f}{\partial x} \right|_{x=x_e}.$$

where:

$J(x_e)$ is the Jacobian matrix of $f(x)$ evaluated at x_e, defined as:

$$J(x_e) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Since $f(x_e) = 0$ at the equilibrium point, the approximation simplifies to:

$$f(x) \approx J(x_e)\delta x.$$

Thus, the dynamics of the system near x_e can be approximated by the linearized system:

$$\dot{\delta x}(t) = J(x_e)\delta x(t),$$

where $\delta x = x - x_e$ represents the deviation from the equilibrium point.

Example: Linear Approximation of a Nonlinear System

Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2, \\ \dot{x}_2 &= -x_2 + x_1 x_2. \end{aligned}$$

Find the linear approximation around the equilibrium point $x_e^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

At $x_e^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, substitute into $f(x)$:

$$f_1(0,0) = -0 + 0^2 = 0,$$

$$f_2(0,0) = -0 + (0)(0) = 0.$$

Thus, $x_e^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is indeed an equilibrium point.
The Jacobian matrix $J(x)$ is given by:

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}.$$

Compute the partial derivatives:

$$\frac{\partial f_1}{\partial x_1} = -1, \quad \frac{\partial f_1}{\partial x_2} = 2x_2, \frac{\partial f_2}{\partial x_1} = x_2, \quad \frac{\partial f_2}{\partial x_2} = -1 + x_1.$$

Thus, the Jacobian matrix is:

$$J(x) = \begin{bmatrix} -1 & 2x_2 \\ x_2 & -1 + x_1 \end{bmatrix}.$$

Evaluate $J(x)$ at $x_e^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The linearized system around $x_e^{(1)}$ is:

$$\dot{\delta x}(t) = J(x_e)\delta x(t),$$

where $\delta x = x - x_e$.

Substitute $J(x_e^{(1)}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$:

$$\dot{\delta x}_1 = -\delta x_1,$$

$$\dot{\delta x}_2 = -\delta x_2.$$

Thus, the dynamics of the linearized system are:

$$\dot{\delta x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \delta x(t).$$

Example: (a) $\dot{x} = x^3$, (b) $\dot{x} = -x^3$

In both cases, linearized systems around $x_e = 0$ are the same: $\dot{x} = 0 \Rightarrow x(t) = x_0$, but NL systems have different behaviors.

Caveats:

1. Only local properties can be determined from the linearization.
2. If $Re(\lambda_i) \leq 0$ with some e-values having $Re(\lambda_i) = 0$, then linearization is inconclusive as a stability test. Higher order terms determine stability.

1.2 Essentially Nonlinear Phenomena Continued

1.2.1 Multiple Isolated Equilibria

Example 1 : $\dot{x} = x(x-2)^2$

$$f(x_e) = 0 \Rightarrow x(x-2)^2 = 0 \Rightarrow \begin{cases} x_e = 0 \\ x_2 = 0 \end{cases}$$

This system has two isolated equilibrium points at 0 and 2.

Example 2 : $\dot{x} = sin(x)$, $x(0) = x_0 \in \mathbb{R}$

$$f(x_e) = \sin x_e = 0 \Rightarrow x_e = k\pi, \qquad k = 0, \pm 1, \pm 2, \cdots.$$

This system has infinite many equilibrium points.

Example 3: Pendulum

$$\ddot{\theta} = -\frac{k}{m}\dot{\theta} - \frac{g}{l}\sin\theta$$

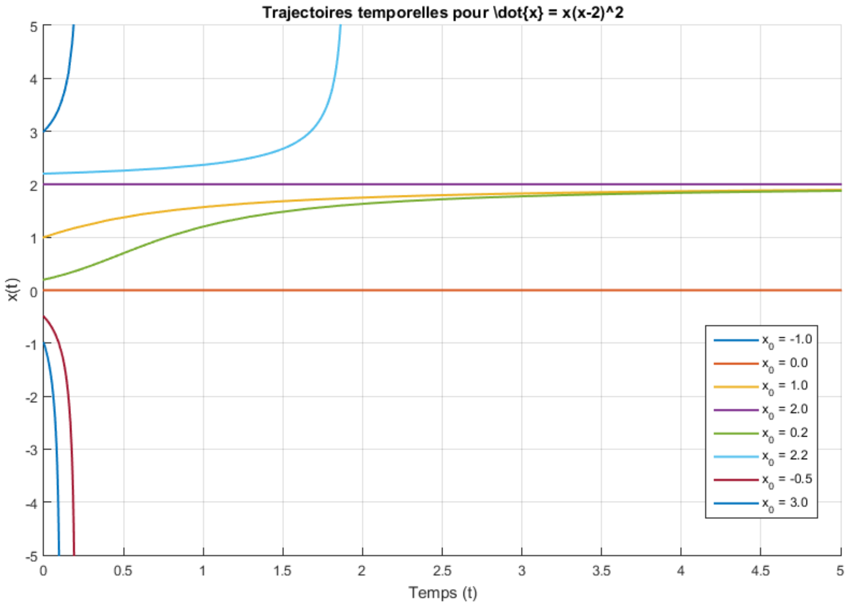


Figure 1.4: Example 2: Multiple Isolated Equilibria

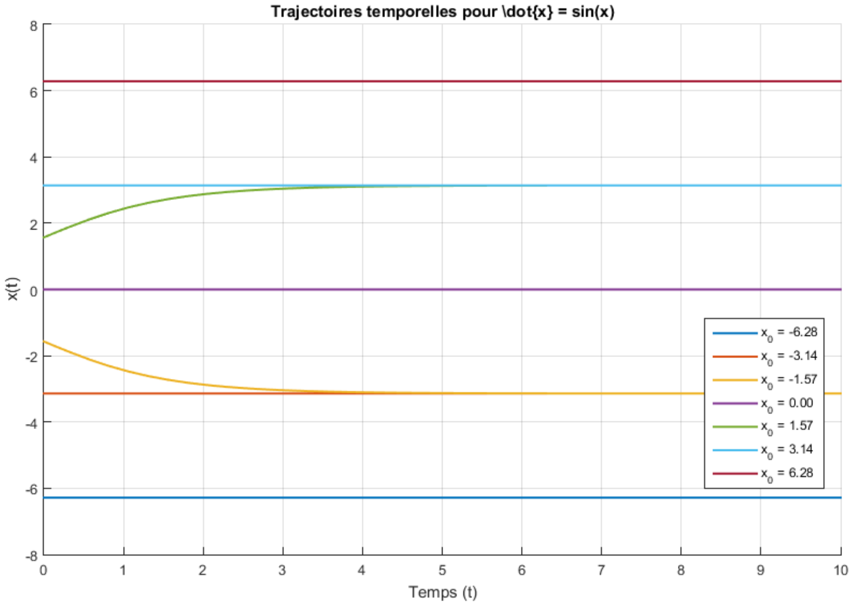


Figure 1.5: Example 2: Multiple Isolated Equilibria

Define $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m} x_2 - \frac{g}{l} \sin x_1 \end{cases}$$

Equilibria : Two isoled EP (0,0) and (π,0).

Linearization :

$$\frac{\delta f}{\delta x} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{l} \end{array} \right] = \left\{ \begin{array}{ll} \left[\begin{array}{cc} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{l} \end{array} \right] & \text{(stable) at } x_1 = 0 \\ \left[\begin{array}{cc} 0 & 1 \\ \frac{g}{l} & -\frac{k}{l} \end{array} \right] & \text{(unstable) at } x_1 = \pi \end{array} \right. \tag{1.4}$$

Stability may depend upon initial conditions

Exemple: The Logistic Equation

The logistic equation is a classic model in population dynamics that describes how a population grows over time under the influence of limited resources. The equation is given by:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right)$$

where:

- $x(t)$ is the population size at time t,
- $r > 0$ is the intrinsic growth rate,
- $K > 0$ is the carrying capacity (the maximum population size the environment can sustain).

Equilibrium Points

To find the equilibrium points, set $\dot{x} = 0$:

$$rx\left(1 - \frac{x}{K}\right) = 0$$

This gives two equilibrium points:

$$x^* = 0 \quad \text{and} \quad x^* = K.$$

Both equilibria are isolated because there are no other solutions to $\dot{x} = 0$ in their neighborhoods.

The stability of each equilibrium point depends on the sign of $\dot{x}$ near the equilibrium and can be analyzed using linearization or qualitative reasoning.

Stability of Equilibrium Points

1. Equilibrium Point $x^* = 0$:

- For small positive x ($0 < x < K$), $\dot{x} = rx\left(1 - \frac{x}{K}\right) > 0$, so the population grows away from $x = 0$.
- For negative x (which is not biologically meaningful but mathematically possible), $\dot{x} < 0$, so trajectories move toward $x = 0$.

Thus, $x^* = 0$ is unstable. Any small initial population will grow away from zero.

2. Equilibrium Point $x^* = K$:

- For $x < K$, $\dot{x} > 0$, so the population grows toward $x = K$.
- For $x > K$, $\dot{x} < 0$, so the population decreases toward $x = K$.

Thus, $x^* = K$ is asymptotically stable. Any initial population near K will converge to K as $t \to \infty$.

1.2.2 Finite Escape Time

In linear case :

$$\dot{x} = \lambda x \qquad \overset{Solution}{\longrightarrow} \qquad x(t) = \exp^{\lambda t} x(0).$$

If $\lambda > 0 \Longrightarrow \lim_{t\to\infty} |x(t)| = +\infty$. Only as $t \to \infty$, $|x(t)| \to \infty$.

In nonlinear case :

Example : $\dot{x} = x^2$, $x(0) = x_0$, $x \in \mathbb{R}$

$$\begin{aligned} \tfrac{dx}{dt} = x^2 \quad &\Rightarrow \quad \textstyle\int_{x_0}^{x(t)} \frac{dx}{x^2} = \int_0^t dt \\ &\Rightarrow \quad -\tfrac{1}{x(t)} + \tfrac{1}{x_0} = t - 0 \end{aligned}$$

$$x(t) = \frac{1}{\frac{1}{x_0} - t}$$

$x_0 > 0 \Rightarrow t \to \frac{1}{x_0} \Rightarrow x(t) \to \infty$.

In the nonlinear equation $\dot{x} = x^2$, the solution $x(t)$ tends to infinity in finite time. Specifically, if $x(0) = x_0$, then $x(t) \to \infty$ as $t \to \frac{1}{x_0}$. This is known as finite escape time, where the system variable grows unbounded in finite time rather than asymptotically as in the linear case.

1.2.3 Limit Cycles

In linear systems, Linear oscillators exhibit a continuum of periodic orbits (closed orbit) :

$$x(t+T) = x(t), \forall t > 0 \quad \text{for somme} \quad t > 0.$$

Every circle is periodic orbit for $\dot{x} = Ax$ where

$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}, \quad (\lambda_{1,2} = \pm j\beta)$$

Example: Harmonic oscillator

$$\underbrace{m\ddot{x}}_{\text{inertial term}} + \underbrace{kx}_{\text{stiffness term}} = 0$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\omega_0 = \sqrt{\frac{k}{m}}$. $(\lambda_{1,2} = \pm j\omega_0)$.

- Amplitude of oscillations depends on initial conditions.
- Can be destroyed by small modelling imperfections.

- The harmonic oscillator has closed orbits but no limit cycles. Limit cycles cannot be generated by LTI systems.
- The linear oscillator is not structurally stable. A stable oscillators must be produced by nonlinear systems.

In nonlinear systems, limit cycles are a fascinating phenomenon where the system exhibits sustained oscillations. A limit cycle is a closed trajectory in the phase space that corresponds to a periodic solution of the system. Nearby trajectories either spiral toward the limit cycle (stable limit cycle) or away from it (unstable limit cycle).

Exemple The Van der Pol Oscillator

A classic example of a nonlinear system that exhibits a limit cycle is the Van der Pol oscillator, which is described by the second-order differential equation:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

where:

- $x(t)$ is the state variable (e.g., position or voltage),
- $\mu > 0$ is a parameter that controls the nonlinearity and damping.

This equation can be rewritten as a system of two first-order differential equations by introducing $y = \dot{x}$:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu(1 - x^2)y - x. \end{aligned}$$

- **Equilibrium Points:** Setting $\dot{x} = 0$ and $\dot{y} = 0$, we find the only equilibrium point: $(x, y) = (0, 0)$.
- **Stability of the Equilibrium:** For small values of x and y, the term $\mu(1 - x^2)y$ acts as positive damping ($\mu > 0$), destabilizing the equilibrium at $(0, 0)$. As a result, the origin is an unstable equilibrium.
- **Limit Cycle:** For $\mu > 0$, the system exhibits a stable limit cycle in the phase plane. This means:
  - Trajectories starting near the origin spiral outward.
  - Trajectories starting far from the origin spiral inward.
  - Eventually, all trajectories converge to a closed curve in the phase plane, representing a periodic solution.
- **Dependence on μ:**
  - When μ is small, the limit cycle is nearly circular, and the oscillations are close to sinusoidal.
  - When μ is large, the limit cycle becomes more distorted, with sharp transitions between slow and fast dynamics (relaxation oscillations).
  - In the (x, y) phase plane:
    - The limit cycle is a closed curve that represents the periodic solution.
    - Inside the limit cycle, trajectories spiral outward toward the cycle.
    - Outside the limit cycle, trajectories spiral inward toward the cycle.

1.2.4 Chaos

Irregular oscillations, never exactly repeating. Behavior of nonlinear systems may be extremely sensive to small changes in initial conditions/input/parameters.

Example : Lorenz system (attractor) derrived by Ed Lorenz in 1963 as a simplified model of convection rolls in the atmosphere.

The lorenz system is a 3rd order system (3 states x,y,z).

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

where:

- x, y, and z represent variables related to the state of the system (e.g., temperature differences and fluid flow rates).
- σ, ρ, and β are parameters:
  - σ: Prandtl number (ratio of momentum diffusivity to thermal diffusivity).
  - ρ: Rayleigh number (a measure of the temperature difference driving convection).
  - β: Geometric factor.

Behavior of the Lorenz System:

- **Chaotic Regime:** For certain parameter values (e.g., $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$), the system exhibits chaotic behavior. Trajectories in phase space do not settle into a fixed point or periodic orbit but instead trace out a complex, fractal-like structure known as the Lorenz attractor.
- **Sensitive Dependence on Initial Conditions:** If you start two trajectories with slightly different initial conditions, they will diverge exponentially over time, illustrating the butterfly effect.

Example: The double pendulum (System is implicit for $l_1 \neq l_2$)

$$\begin{cases} (m_1 + m_1)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2)\sin\theta_1 & = & 0 \\ m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 & = & 0 \end{cases}$$

1.2.5 Bifurcation:

A bifurcation is an abrupt change in qualitative behavior as a parameter is varied.
Examples : creation (or death) of equilibrium points (or limit cycles) and/or change of their stability properties.

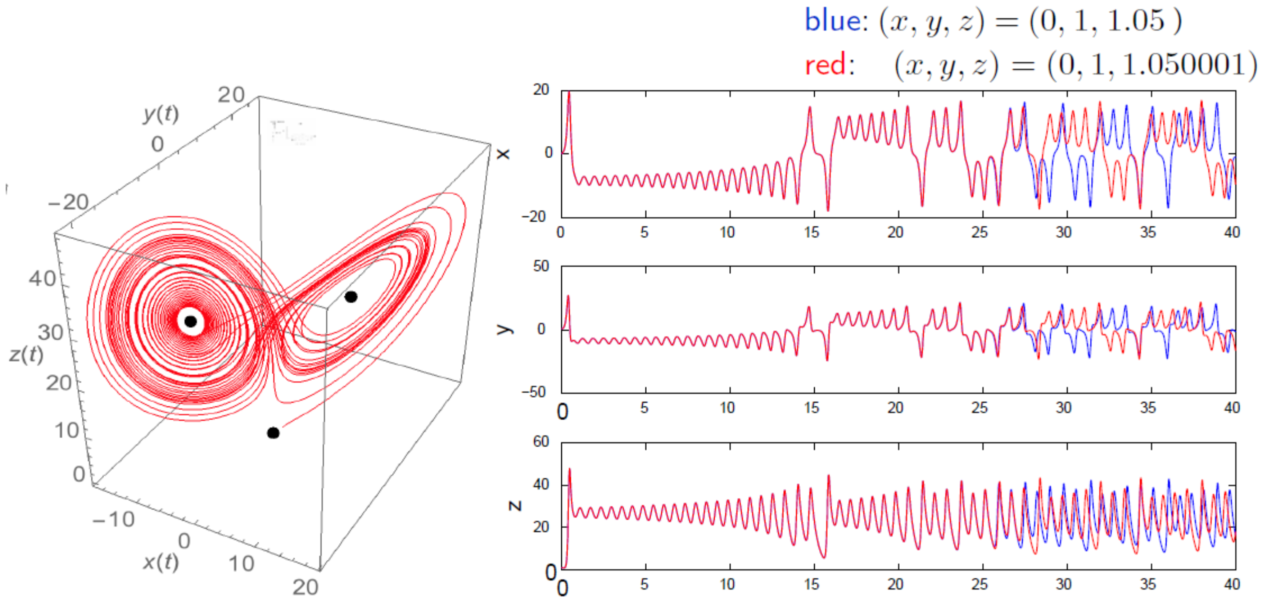


Figure 1.6: Lorenz System

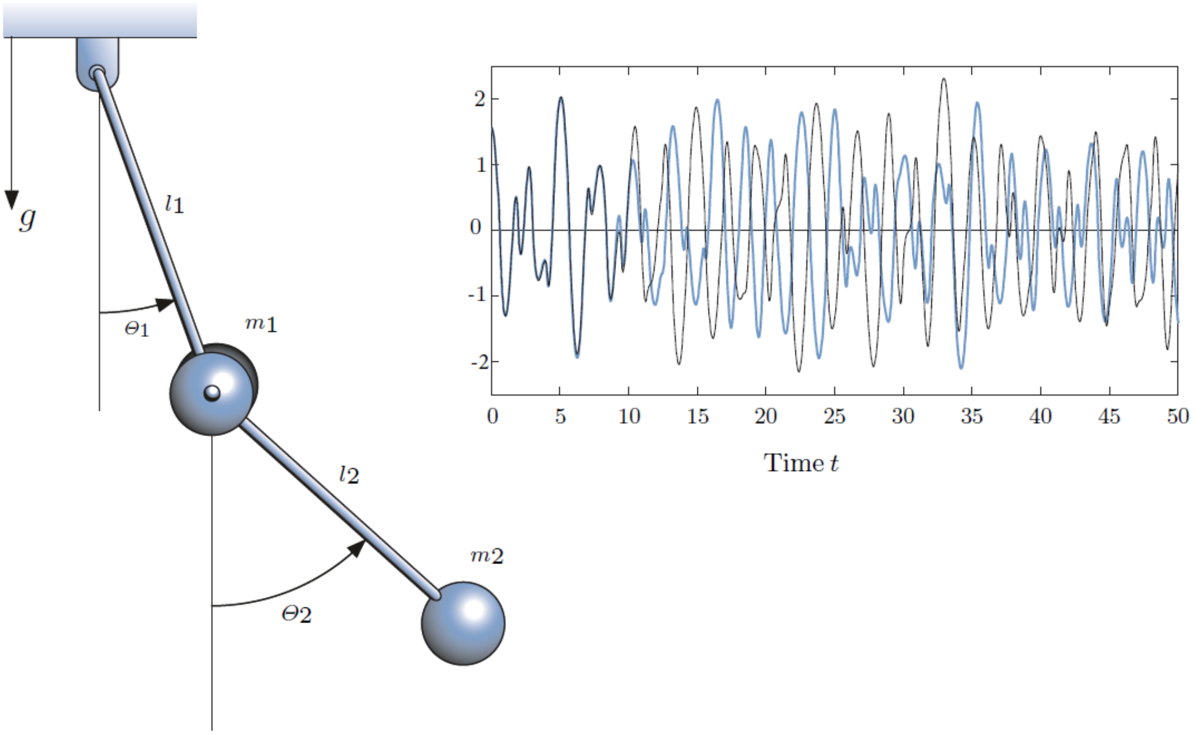


Figure 1.7: The double pendulum

Fold bifurcation: 1st order system

Example : $\dot{x} = \mu - x^2$,

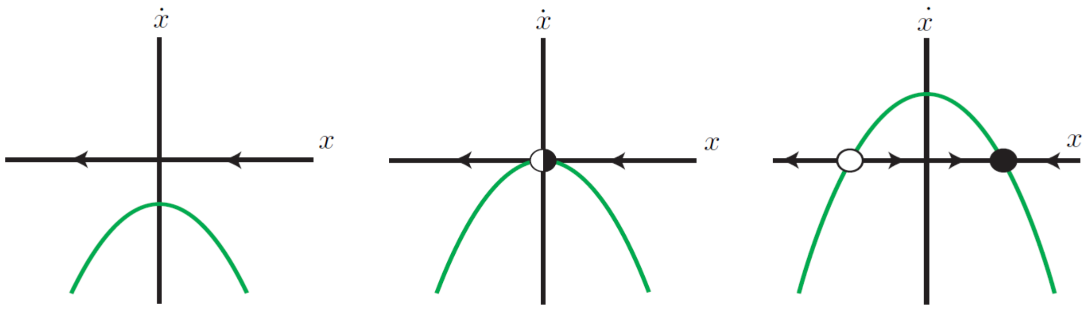


Figure 1.8: Fold bifurcation

- Equilibrium points : $\bar{x} = \begin{cases} \pm\sqrt{\mu} & \mu > 0 \quad \text{one stable equilibrium} \\ & \qquad\quad\;\; \text{and one unstable equilibrium} \\ 0 & \mu = 0 \quad \text{single equilibrium (called a saddle)} \\ \text{none} & \mu < 0 \quad \text{no equilibria} \end{cases}$

$\mu_c = 0$ is the critical value of parameter μ which represents boundary between "no equilibrium points" and the presence of equilibrium points.

Creation/destruction of fixed points is called **saddle node bifurcation**

Linearization :

$$\left.\frac{\delta f}{\delta x}\right|_{\bar{x}} = 2\bar{x} = \begin{cases} 2\sqrt{\mu} & \text{unstable} \\ -2\sqrt{\mu} & \text{stable} \end{cases} \qquad \mu > 0.$$

Note: $A_c = \left.\frac{\delta f}{\delta x}\right|_{\bar{x}_c = \bar{x}(\mu_c)} = 0 \rightarrow$ linearization disappears, no information about stability of the system.

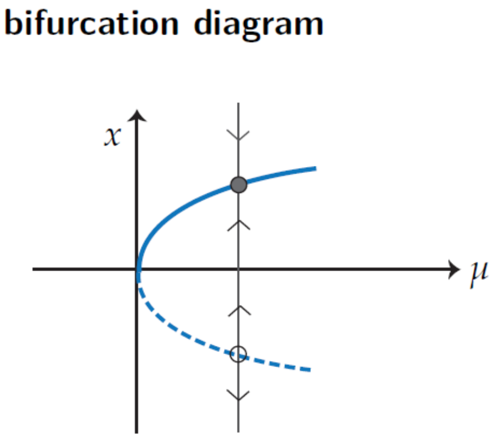


Figure 1.9: Bifurcation diagram

Transcritical bifurcation:

Example : $\dot{x} = \mu x - x^2, \quad x(t) \in \mathbb{R}$

- Equilibrium points :

$$f(\bar{x}) = 0 \Rightarrow \bar{x}(\mu - \bar{x}) = 0 \Rightarrow \begin{cases} \bar{x} = 0. \\ \bar{x} = \mu. \end{cases}$$

- Linearization :

$$\frac{\delta f}{\delta x} = \mu - 2\bar{x} = \begin{cases} \mu & \text{if} \quad \bar{x} = 0 \\ -\mu & \text{if} \quad \bar{x} = \mu \end{cases}$$

$\mu < 0 : \bar{x} = 0$ is stable, $\bar{x} = \mu$ is unstable.
$\mu > 0 : \bar{x} = 0$ is unstable, $\bar{x} = \mu$ is stable.

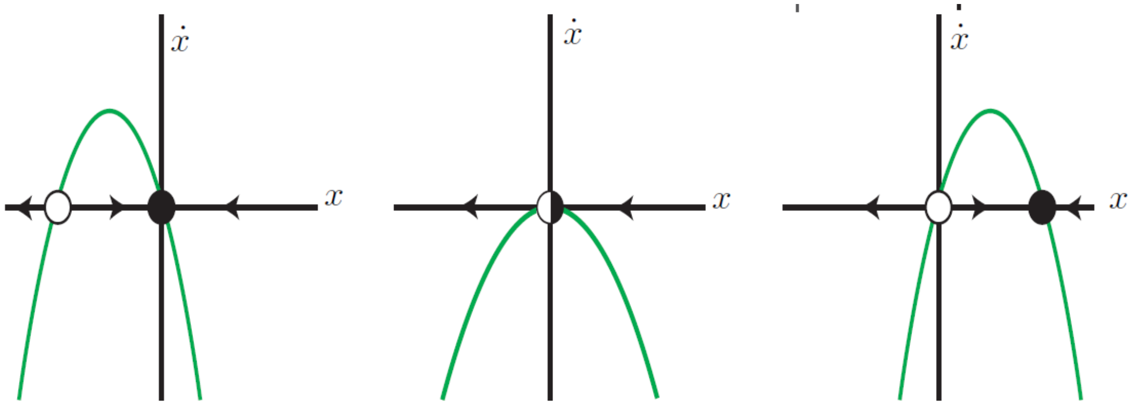


Figure 1.10: Transcritical bifurcation

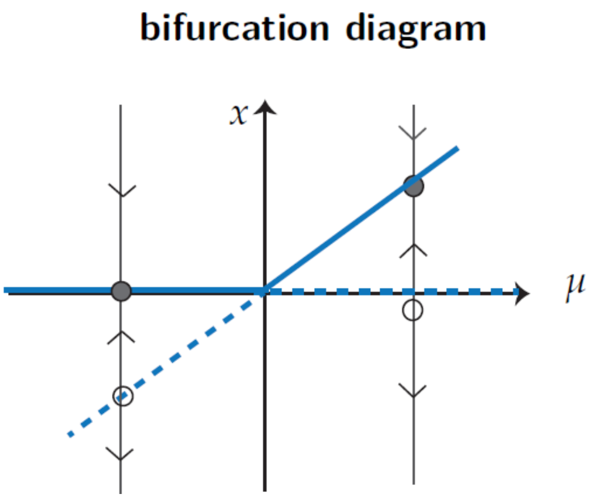


Figure 1.11: Transcritical bifurcation diagram

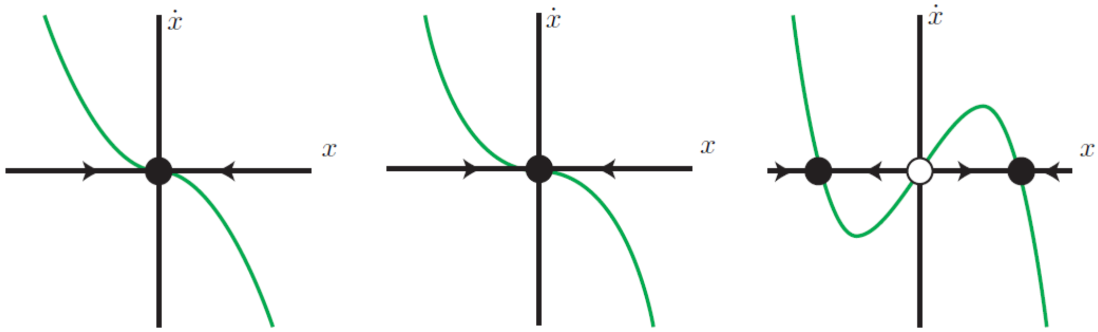
Bifurcation : Pichfork bifurcation

Pitchfork Bifurcation- 2 types : supercritical Pitchfork and subcritical pitchfork

Example 1:

$$\dot{x} = \mu x - x^3 \quad \text{supercritical}$$

- Equilibrium points : $f(\bar{x}) = 0 \Rightarrow \bar{x}(\mu - \bar{x}^2) = 0 \Rightarrow \bar{x} = \begin{cases} 0 \\ \pm\sqrt{\mu}, & \mu > 0 \end{cases}$



2 equilibrium points emerge when we increase μ.

Figure 1.12: Pichfork bifurcation

Example 2: $\dot{x} = \mu x + x^3$, subcritical pitchfork.

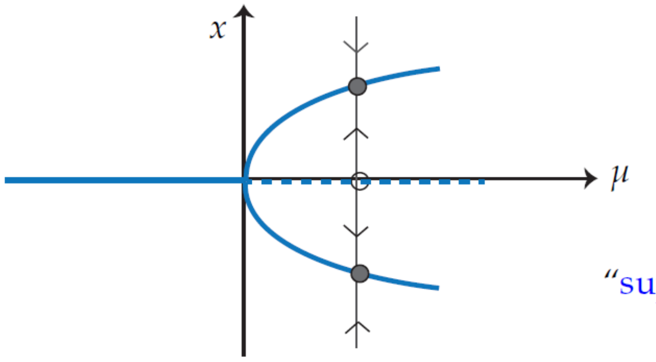
- Equilibrium points :

$$f(\bar{x}) = 0 \Rightarrow \bar{x}(\mu - \bar{x}^2) = 0 \Rightarrow \bar{x} = \begin{cases} 0 \\ \pm\sqrt{-\mu}, & \mu < 0 \end{cases}$$

- Linearization :

$$\begin{aligned} \left.\frac{\delta f}{\delta x}\right|_{\bar{x}=0} &= \mu \\ \left.\frac{\delta f}{\delta x}\right|_{\bar{x}=\pm\sqrt{-\mu}} &= -2\mu \end{aligned}$$

bifurcation diagram



“supercritical pitchfork"

Figure 1.13: Supercritical bifurcation diagram

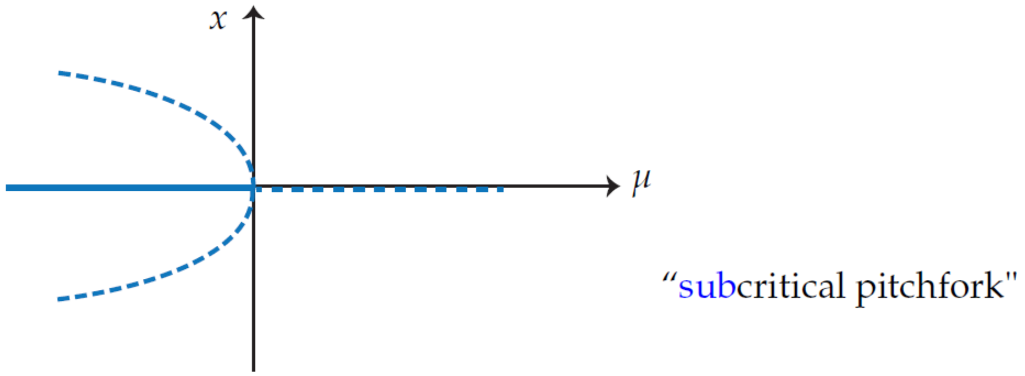


Figure 1.14: SuBcritical bifurcation diagram

Chapter 2

Second Order Systems

2.1 Concept of Phase Plane

A second-order autonomous system is represented by two scalar differential equations

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \tag{2.1}$$

Let $x(t) = (x_1(t), x_2(t))$ be the solution of (2.1) that starts at a certain initial state $x_0 = (x_{10}, x_{20})$.

- f(.) is called a **vector field**

The set of points $\{(t, x_1(t), x_2(t)); t \in \mathbb{R}\}$ with (x_1, x_2) a solution of (2.1) (and $x_1(t_0) = x_{10}$ and $x_2(t_0) = x_{20}$ for some t_0) is called the **trajectory** or **solution curve** (through (x_{10}, x_{20})).

The set of points $\{(x_1(t), x_2(t)); t \in \mathbb{R}\}$ with (x_1, x_2) a solution of (2.1) (and $x_1(t_0) = x_{10}$ and $x_2(t_0) = x_{20}$ for some t_0) is called the $fOrbit$ or **Phase Curve** (through (x_{10}, x_{20})).

- An orbit that forms a closed curve is called a **closed orbit**.
- The family of all trajectories of a dynamical system is called **the phase portrait.**

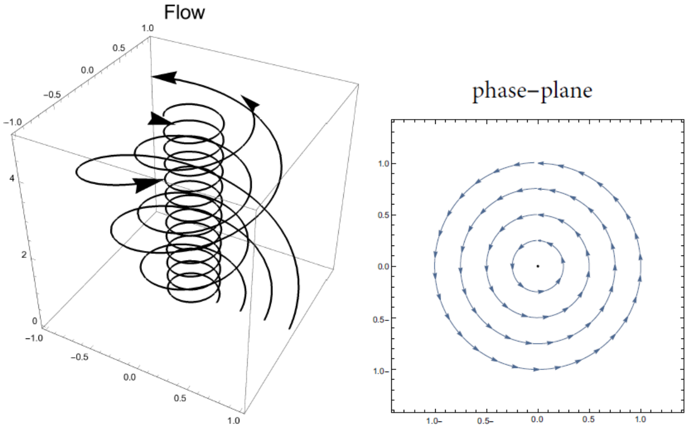


Figure 2.1: Concept of Phase Plane

2.1.1 Vector fields and Orbits

Example: Uncoupled system $\begin{cases} \dot{x} &= 2x &= f_x(x,y) \\ \dot{y} &= -3y &= f_y(x,y) \end{cases}$

- $\mathbf{f}$ is a **vector field** in $\mathbb{R}^2$.
- Solution : $(x_0 e^{2t}, y_0 e^{-3t})$

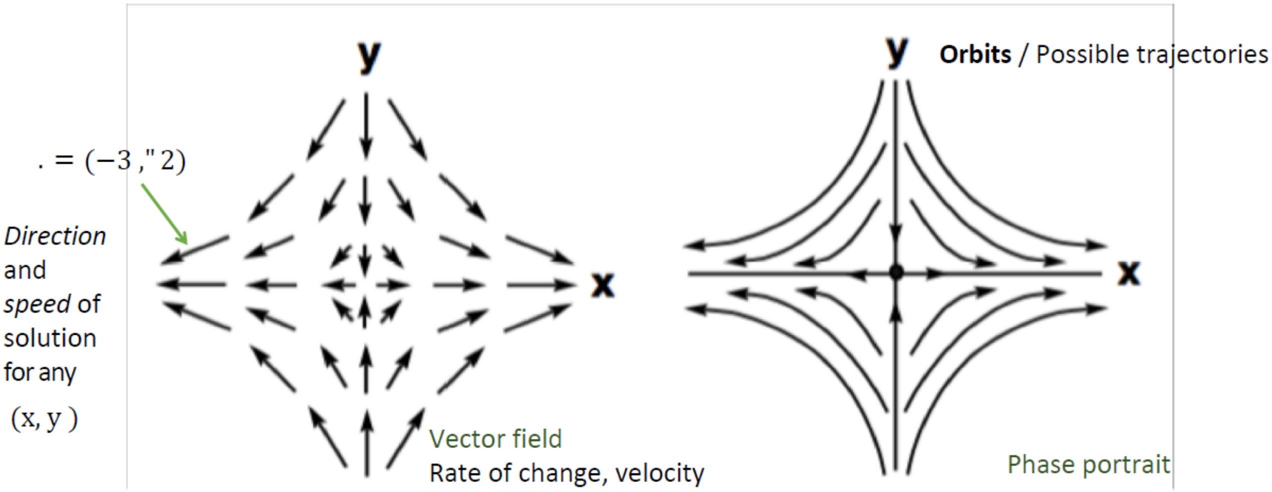


Figure 2.2: Vector fields and Orbits

Example2:

$$\begin{cases} \dot{x} = y = f_x(x,y) \\ \dot{y} = -x - y^2 = f_y(x,y) \end{cases} \qquad \begin{array}{l} (x,y) \rightarrow f(x, -x-y^2) \\ (1,1) \rightarrow (1,-2) \end{array}$$

2.1.2 Phase plan analysis

Problem

When $x(t) \in \mathbb{R}^2$, study state trajectories around an equilibrium state

2.1.3 Behavior of Linear second Order Systems

Consider the following linear system

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.2}$$

$(a,b,c,d) \in \mathbb{R}$. Eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of A satisfy

$$A\mathbf{v}_i = \mathbf{v}_i \lambda_i$$

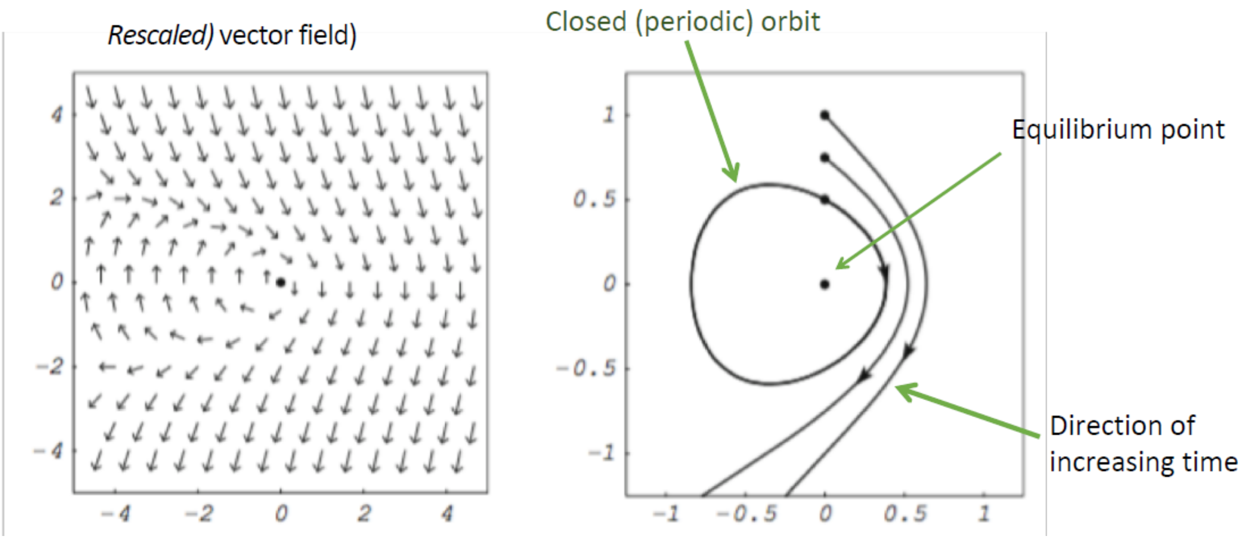
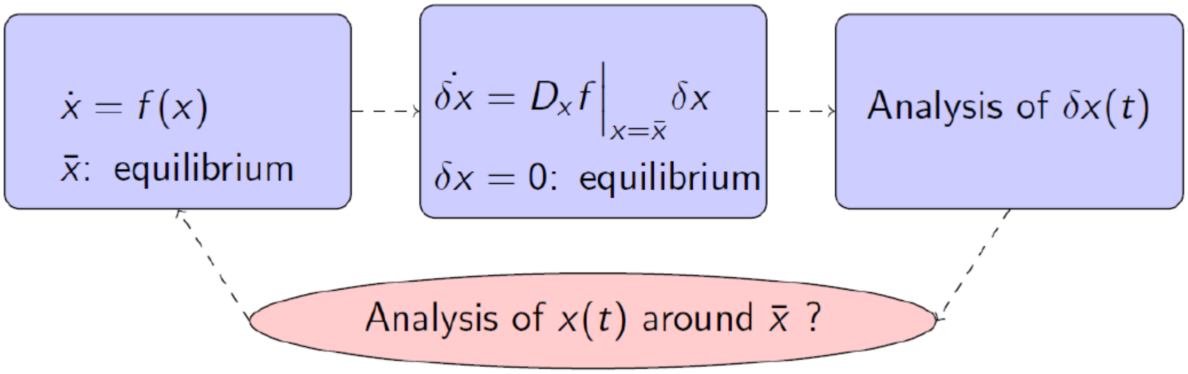


Figure 2.3: Vector fields



The matrix A is then directly diagonalizable

$$AV = V\Lambda \implies A = V\Lambda V^{-1}$$

Λ Change of coordinates: $z(t) = T^{-1}x(t)$, $T \in \mathbb{R}^{2\times 2}$ invertible.

$$\dot{z}(t) = T^{-1}\dot{x}(t) = T^{-1}Ax(t) = T^{-1}ATz(t) = Jz(t)$$

The system $\dot{z} = Jz$ is equivalent to the system $\dot{x} = Ax$.

Remark

A and $J = T^{-1}AT$ are similar $\implies$ they have the same eigenvalues

- One can always choose T such that J is in real Jordan form

  1. the new coordinates are called normal

The solution of (2.2) for an initial condition x_0 is given by

$$x(t) = Te^{J_r t}T^{-1}x_0$$

where J_r is the real Jordan form of A and M is a real nonsingular matrix such that

$$J_r = T^{-1}AT$$

The fixed point $x_e = 0$ and the phase portrait near $x_e = 0$ are fully determined by the eigenvalues and eigenvectors of system matrix A.

There are three possible Jordan forms for A:

- Different real eigenvalues.
- Equal real eigenvalues.
- Complex conjugate.

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix} \qquad \begin{bmatrix} \alpha & \beta \\ -\beta & \lambda \end{bmatrix}$$

where $k = 0$ or 1.

- In addition, we need to consider the case where at least one of the eigenvalues is zero.

Case 1: $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\lambda_1 \in \mathbb{R}$, and independent eigenvectors
In this case

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2 \Longrightarrow T = [\mathbf{v}_1 \quad \mathbf{v}_2]$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the real eigenvectors of A associated with λ_1 and λ_2, respectively.

The change of coordinate $z = T^{-1}x$, transforms the system into two decoupled first-order differential equations, i.e.,

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{z}_2 = \lambda_1 z_2$$

with solution

$$z_1(t) = z_{10}e^{\lambda_1 t}, \quad z_2(t) = z_{20}e^{\lambda_2 t} \Longrightarrow z_2(t) = \frac{z_{20}}{(z_{10})^{\lambda_2/\lambda_1}} z_1^{\lambda_2/\lambda_1}$$

Case 1a: $\lambda_1 < 0$ and $\lambda_2 < 0$

The origin is called **stable node**

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= -6x - 2y \\ \dot{y} &= -2x - 9y \end{aligned}$$

To find the eigenvalues and eigenvectors of the given system, we can rewrite it in matrix form and solve the corresponding eigenvalue problem.

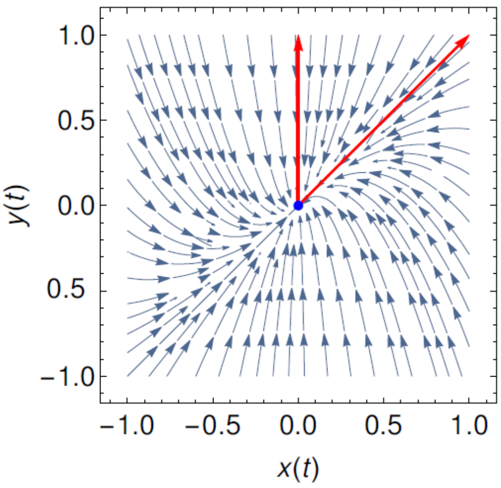


Figure 2.4: stable node

We can express this system as a matrix equation:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -6 & -2 \\ -2 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The coefficient matrix A is:

$$A = \begin{pmatrix} -6 & -2 \\ -2 & -9 \end{pmatrix}$$

1. The eigenvalues λ are found by solving the characteristic equation:

$$\det(A - \lambda I) = 0$$

   where I is the identity matrix.

$$A - \lambda I = \begin{pmatrix} -6-\lambda & -2 \\ -2 & -9-\lambda \end{pmatrix}$$

   Now compute the determinant:

$$\det(A - \lambda I) = \det \begin{pmatrix} -6-\lambda & -2 \\ -2 & -9-\lambda \end{pmatrix}$$

$$= (-6-\lambda)(-9-\lambda) - (-2)(-2)$$

$$= (\lambda+6)(\lambda+9) - 4$$

$$= \lambda^2 + 15\lambda + 54 - 4$$

$$= \lambda^2 + 15\lambda + 50$$

We now solve the quadratic equation:

$$\lambda^2 + 15\lambda + 50 = 0$$

Using the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = 15$, and $c = 50$. First, compute the discriminant:

$$\Delta = b^2 - 4ac = 15^2 - 4(1)(50) = 225 - 200 = 25$$

Now compute the eigenvalues:

$$\lambda = \frac{-15 \pm \sqrt{25}}{2} = \frac{-15 \pm 5}{2}$$

So we have two solutions for λ:

$$\lambda_1 = \frac{-15 + 5}{2} = -5 \quad \text{and} \quad \lambda_2 = \frac{-15 - 5}{2} = -10$$

2. To find the eigenvectors, we solve the system $(A - \lambda I)\mathbf{v} = 0$ for each eigenvalue λ.

    - For $\lambda_1 = -5$: We need to solve:

    $$(A + 5I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

    $$A + 5I = \begin{pmatrix} -6+5 & -2 \\ -2 & -9+5 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}$$

    Now, solve the system:

    $$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

    This gives the equations:

    $$-v_1 - 2v_2 = 0 \quad \text{and} \quad -2v_1 - 4v_2 = 0$$

    From the first equation, we get $v_1 = -2v_2$. Therefore, the eigenvector corresponding to $\lambda_1 = -5$ is:

    $$\mathbf{v_1} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

- For $\lambda_2 = -10$: We need to solve:

$$(A + 10I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$A + 10I = \begin{pmatrix} -6+10 & -2 \\ -2 & -9+10 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

Now, solve the system:

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the equations:

$$4v_1 - 2v_2 = 0 \quad \text{and} \quad -2v_1 + v_2 = 0$$

From the second equation, we get $v_1 = \frac{1}{2}v_2$. Therefore, the eigenvector corresponding to $\lambda_2 = -10$ is:

$$\mathbf{v_2} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

Conclusion: The eigenvalues of the system are $\lambda_1 = -5$ and $\lambda_2 = -10$, and the corresponding eigenvectors are:

- For $\lambda_1 = -5$, the eigenvector is $\mathbf{v_1} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. - For $\lambda_2 = -10$, the eigenvector is $\mathbf{v_2} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

Case 1b: $\lambda_1 > 0$ and $\lambda_2 > 0$

The origin is called **Unstable Node**

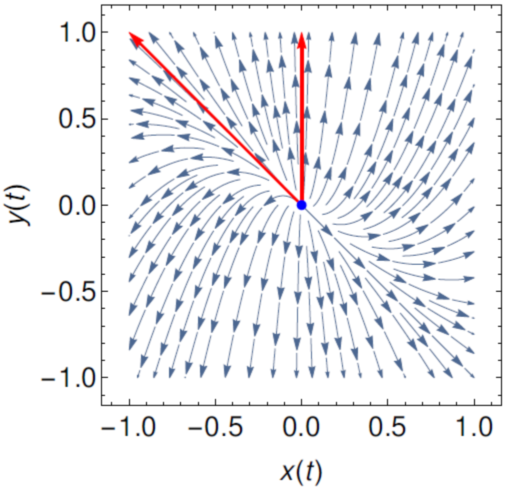


Figure 2.5: Unstable Node

Example: Consider the linear system

$$\begin{aligned}\dot{x} &= x - 2y \\ \dot{y} &= x + 4y\end{aligned}$$

This can be written in matrix form as:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & -2\\ 1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

So, the coefficient matrix A is:

$$A = \begin{pmatrix}1 & -2\\ 1 & 4\end{pmatrix}$$

1. The eigenvalues λ are found by solving the characteristic equation:

$$\det(A - \lambda I) = 0$$

where I is the identity matrix.

$$A - \lambda I = \begin{pmatrix}1-\lambda & -2\\ 1 & 4-\lambda\end{pmatrix}$$

Now compute the determinant:

$$\det(A - \lambda I) = \det\begin{pmatrix}1-\lambda & -2\\ 1 & 4-\lambda\end{pmatrix}$$

$$= (1-\lambda)(4-\lambda) - (-2)(1)$$

$$= (1-\lambda)(4-\lambda) + 2$$

$$= 4 - \lambda - 4\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 5\lambda + 6$$

We now solve the quadratic equation:

$$\lambda^2 - 5\lambda + 6 = 0$$

Using the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = -5$, and $c = 6$. First, compute the discriminant:

$$\Delta = (-5)^2 - 4(1)(6) = 25 - 24 = 1$$

Now compute the eigenvalues:

$$\lambda = \frac{5 \pm \sqrt{1}}{2} = \frac{5 \pm 1}{2}$$

So we have two solutions for λ:

$$\lambda_1 = \frac{5+1}{2} = 3 \quad \text{and} \quad \lambda_2 = \frac{5-1}{2} = 2$$

2. To find the eigenvectors, we solve the system $(A - \lambda I)\mathbf{v} = 0$ for each eigenvalue λ.

   - For $\lambda_1 = 3$:
     We need to solve:

     $$(A - 3I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

     $$A - 3I = \begin{pmatrix} 1-3 & -2 \\ 1 & 4-3 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}$$

     Now, solve the system:

     $$\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

     This gives the equations:

     $$-2v_1 - 2v_2 = 0 \quad \text{and} \quad v_1 + v_2 = 0$$

     From the second equation, we get $v_1 = -v_2$. Therefore, the eigenvector corresponding to $\lambda_1 = 3$ is:

     $$\mathbf{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

   - For $\lambda_2 = 2$:
     We need to solve:

     $$(A - 2I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

     $$A - 2I = \begin{pmatrix} 1-2 & -2 \\ 1 & 4-2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$$

     Now, solve the system:

     $$\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the equations:

$$-v_1 - 2v_2 = 0 \quad \text{and} \quad v_1 + 2v_2 = 0$$

From the second equation, we get $v_1 = -2v_2$. Therefore, the eigenvector corresponding to $\lambda_2 = 2$ is:

$$\mathbf{v_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

- Conclusion:
  The eigenvalues of the system are $\lambda_1 = 3$ and $\lambda_2 = 2$, and the corresponding eigenvectors are:
  - For $\lambda_1 = 3$, the eigenvector is $\mathbf{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. - For $\lambda_2 = 2$, the eigenvector is $\mathbf{v_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Case 1d: $\lambda_1{=}\lambda_2$

The origin is called **stable/unstable degenerate node**

- **Stable degenerate node**: $\lambda_1{=}\lambda_2 < 0$

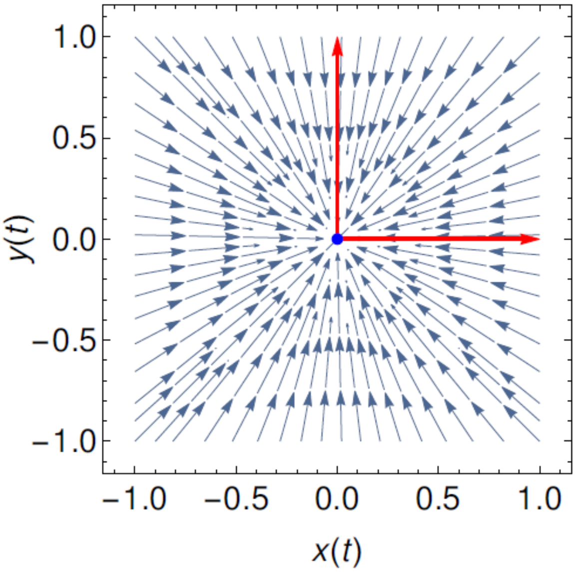


Figure 2.6: Stable degenerate node

- **Unstable degenerate node**: $\lambda_1{=}\lambda_2 > 0$

Case 1b: $\lambda_1 < 0 < \lambda_2$

The origin is called **saddle**

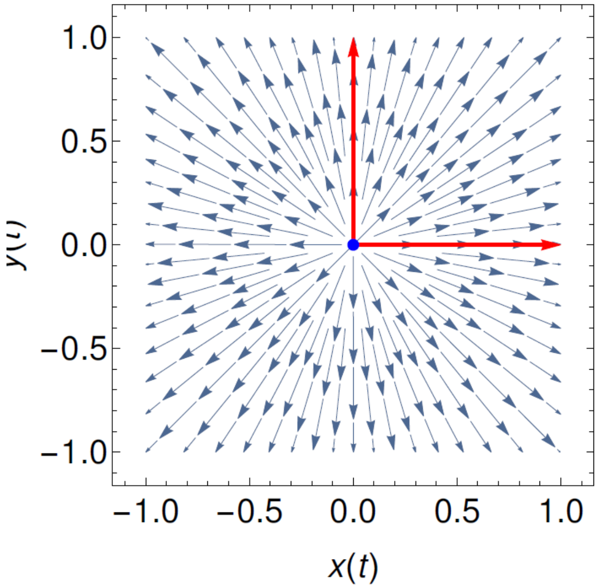


Figure 2.7: Unstable degenerate node

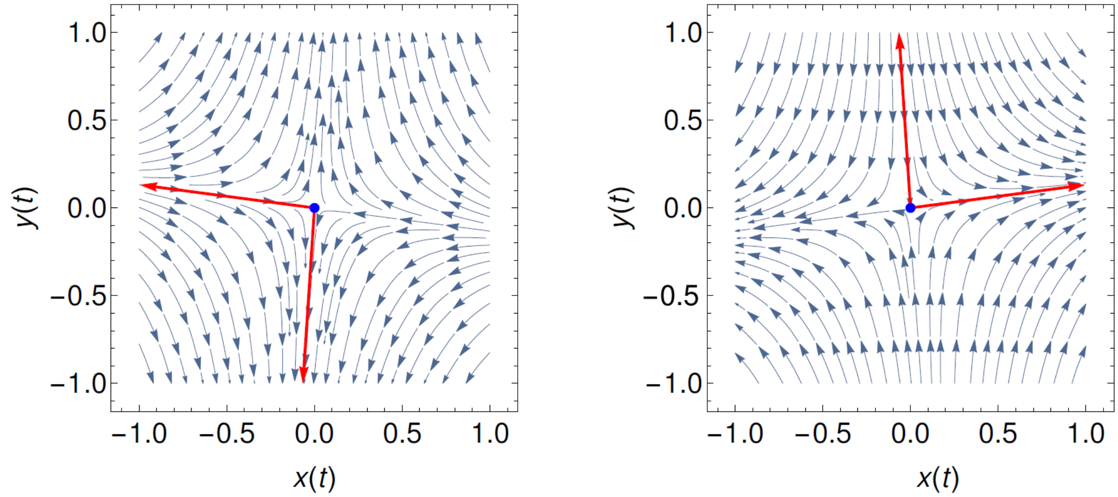


Figure 2.8: The origin is called **saddle**

Example: Consider the linear system

$$\begin{aligned}\dot{x} &= 3x + 4y \\ \dot{y} &= x\end{aligned}$$

- eigenvalues : $\lambda_1 = 4$; $\lambda_2 = -1 \Longrightarrow \mathit{Saddle}$
- eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$

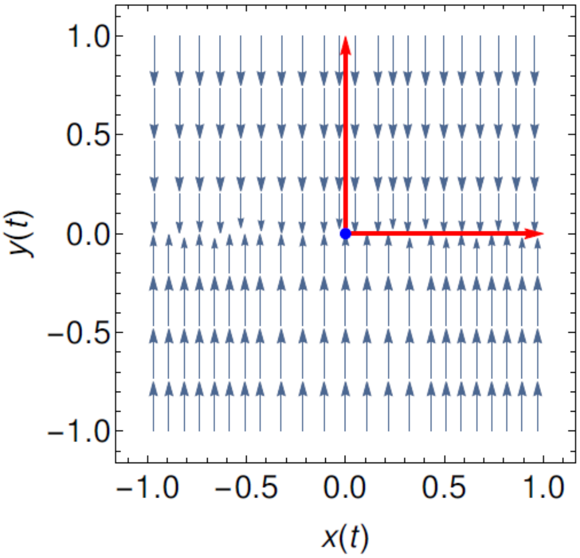


Figure 2.9: Case 1 : degenerate saddle

Case 1e: degenerate saddle

- $\lambda_1 < \lambda_2 = 0 \rightarrow$ all states on the z_2 axis are equilibrium states.
- $0 = \lambda_1 < \lambda_2 \rightarrow$ all states on the z_1 axis are equilibrium states.

- $\lambda_1 < \lambda_2 = 0$

- $0 = \lambda_1 < \lambda_2$

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= 3x - y \\ \dot{y} &= -3x + y \end{aligned}$$

This can be written in matrix form as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So, the coefficient matrix A is:

$$A = \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}$$

1. The eigenvalues λ are found by solving the characteristic equation:

$$\det(A - \lambda I) = 0$$

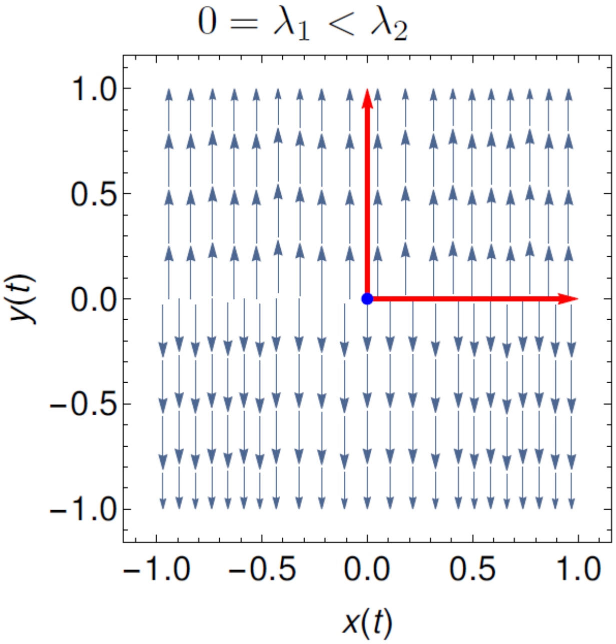


Figure 2.10: Case 2 :degenerate saddle

$$A - \lambda I = \begin{pmatrix} 3-\lambda & -1 \\ -3 & 1-\lambda \end{pmatrix}$$

Now compute the determinant:

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & -1 \\ -3 & 1-\lambda \end{pmatrix}$$

$$= (3-\lambda)(1-\lambda) - (-1)(-3)$$

$$= (3-\lambda)(1-\lambda) - 3$$

$$= 3 - \lambda - 3\lambda + \lambda^2 - 3$$

$$= \lambda^2 - 4\lambda$$

We now solve the quadratic equation:

$$\lambda^2 - 4\lambda = 0$$

Factor the equation:

$$\lambda(\lambda - 4) = 0$$

So, the eigenvalues are:

$$\lambda_1 = 0, \quad \lambda_2 = 4$$

2. To find the eigenvectors, we solve the system $(A - \lambda I)\mathbf{v} = 0$ for each eigenvalue λ.
   - For $\lambda_1 = 0$:
     We need to solve:

     $$(A - 0I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

     $$A = \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}$$

     Now, solve the system:

     $$\begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

     This gives the equations:

     $$3v_1 - v_2 = 0 \quad \text{and} \quad -3v_1 + v_2 = 0$$

     From the first equation, we get $v_2 = 3v_1$. Therefore, the eigenvector corresponding to $\lambda_1 = 0$ is:

     $$\mathbf{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

   - For $\lambda_2 = 4$:
     We need to solve:

     $$(A - 4I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

     $$A - 4I = \begin{pmatrix} 3-4 & -1 \\ -3 & 1-4 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -3 & -3 \end{pmatrix}$$

     Now, solve the system:

     $$\begin{pmatrix} -1 & -1 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

     This gives the equations:

     $$-v_1 - v_2 = 0 \quad \text{and} \quad -3v_1 - 3v_2 = 0$$

     From the first equation, we get $v_1 = -v_2$. Therefore, the eigenvector corresponding to $\lambda_2 = 4$ is:

     $$\mathbf{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

- Conclusion:

  The eigenvalues of the system are $\lambda_1 = 0$ and $\lambda_2 = 4$, and the corresponding eigenvectors are:

  - For $\lambda_1 = 0$, the eigenvector is $\mathbf{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. - For $\lambda_2 = 4$, the eigenvector is $\mathbf{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Example: Consider the linear system

$$\begin{array}{rcl} \dot{x} & = & x - 2y \\ \dot{y} & = & -2x - 4y \end{array}$$

- eigenvalues : $\lambda_1 = -5$; $\lambda_2 = 0 \Longrightarrow$ *Degenerate Sink*
- eigen-vectors : $\vec{v_1} = \left(\begin{array}{c} 0.5 \\ 1 \end{array} \right), \quad \vec{v_2} = \left(\begin{array}{c} -2 \\ 1 \end{array} \right)$

Case 2: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $\lambda \in \mathbb{R}$, One can show that the state trajectories are given by

$$\begin{array}{l} \dot{z}_1 = z_{10}e^{\lambda t} + z_{20}te^{\lambda t} \\ \dot{z}_2 = z_{20}e^{\lambda t} \end{array} \tag{2.3}$$

Assume $z_{20} \neq 0$. If $\lambda \neq 0$, from (2.3-b-) one gets

$$e^{\lambda t} = \frac{z_2(t)}{z_{20}} \Longrightarrow t = \frac{1}{\lambda} \ln \left(\frac{z_2(t)}{z_{20}} \right)$$

and using (2.3-a-) one obtains

$$z_1(t) = z_{10}\frac{z_2(t)}{z_{20}} + \frac{1}{\lambda} \ln \left(\frac{z_2(t)}{z_{20}} \right) z_2(t)$$

Case 2: $\lambda \neq 0$

The origin is called **stable/unstable improper node**

- Only the z_1 axis is invariant.

Case 3: Complex conjugate eigenvalues $\lambda_{1,2} = \alpha \pm j\beta \in \mathbb{C}$
Let $\mathbf{v}_1 = u + jv$, $\mathbf{v}_2 = u - jv$ be the eigenvectors associated to the eigenvalues $\lambda_1 = \alpha + j\beta$, $\lambda_2 = \alpha - j\beta$. One has

$$A(u + jv) = (\alpha + j\beta)(u + jv) \qquad A(u - jv) = (\alpha - j\beta)(u - jv)$$

Summing and subtracting: $Au = \alpha u - \beta v \qquad Av = \beta u + \alpha v$

$$\Rightarrow T = [\mathbf{v}_1 \quad \mathbf{v}_2]$$

$\mathbf{v}_1$ and $\mathbf{v}_2$ are the real eigenvectors of A associated with λ_1 and λ_2, respectively.
The change of coordinate $z = T^{-1}x$ transforms the system into the form

$$\dot{z}_1 = \alpha z_1 - \beta z_2, \qquad \dot{z}_2 = \beta z_1 + \alpha z_2$$

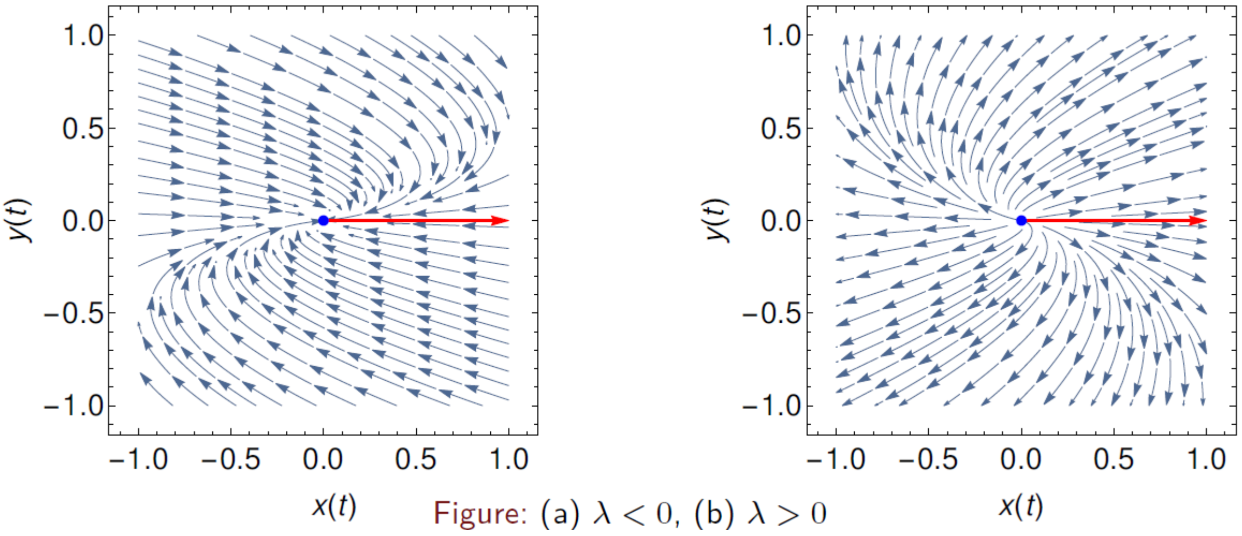


Figure 2.11: Stable/unstable improper node

Defining the change of coordinates

$$r = \sqrt{z_1^2 + z_2^2} \qquad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

we can write the dynamic equations in polar coordinates as

$$\dot{r} = \alpha r, \qquad \dot{\theta} = \beta$$

with solution

$$r(t) = r_0 e^{\alpha t}, \qquad \theta(t) = \theta_0 + \beta t$$

Case 3a: $\alpha < 0$

The origin is called **Stable Focus**

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= -2x + y \\ \dot{y} &= -2x - y \end{aligned}$$

This can be written in matrix form as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

1. To find the eigenvalues, we need to solve the characteristic equation:

   $$\det(A - \lambda I) = 0$$

   The matrix $A - \lambda I$ is:

   $$A - \lambda I = \begin{bmatrix} -2 - \lambda & 1 \\ -2 & -1 - \lambda \end{bmatrix}$$

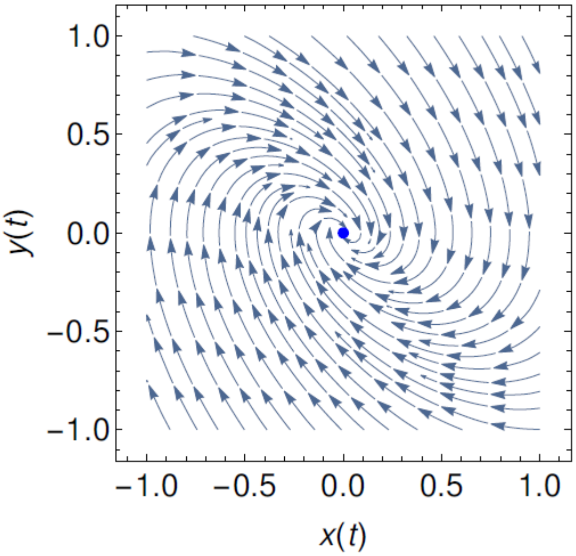


Figure 2.12: Stable Focus

Now, compute the determinant:

$$\begin{aligned}\det(A-\lambda I) &= (-2-\lambda)(-1-\lambda)-(-2)(1)\\ &= (\lambda+2)(\lambda+1)+2\\ &= \lambda^2+3\lambda+4\end{aligned}$$

Set this equal to zero:

$$\lambda^2+3\lambda+4=0$$

Now, solve for λ using the quadratic formula:

$$\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

For the quadratic equation $a=1$, $b=3$, and $c=4$, we get:

$$\lambda=\frac{-3\pm i\sqrt{7}}{2}$$

So, the eigenvalues are:

$$\lambda_1=\frac{-3+i\sqrt{7}}{2},\quad \lambda_2=\frac{-3-i\sqrt{7}}{2}$$

These are complex eigenvalues with a negative real part $\left(\mathrm{Re}(\lambda)=-\frac{3}{2}\right)$.

2. Find the eigenvectors

- Eigenvector for $\lambda_1 = \frac{-3+i\sqrt{7}}{2}$
  We substitute $\lambda_1 = \frac{-3+i\sqrt{7}}{2}$ into $A - \lambda_1 I$:

  $$A - \lambda_1 I = \begin{bmatrix} -2 - \frac{-3+i\sqrt{7}}{2} & 1 \\ -2 & -1 - \frac{-3+i\sqrt{7}}{2} \end{bmatrix}$$

  Simplifying the matrix, we get the following system of equations. However, it's easier to solve this using standard linear algebra techniques or numerical methods. The eigenvector corresponding to λ_1 will be of the form:

  $$v_1 = \begin{bmatrix} 1 \\ \frac{1+i\sqrt{7}}{2} \end{bmatrix}$$

- Eigenvector for $\lambda_2 = \frac{-3-i\sqrt{7}}{2}$
  Similarly, the eigenvector corresponding to $\lambda_2 = \frac{-3-i\sqrt{7}}{2}$ is:

  $$v_2 = \begin{bmatrix} 1 \\ \frac{1-i\sqrt{7}}{2} \end{bmatrix}$$

Case 3a: $\alpha > 0$

The origin is called **Unstable Focus**

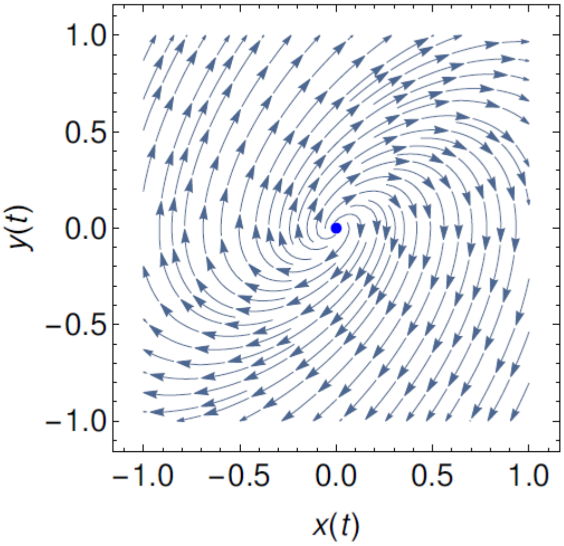


Figure 2.13: Unstable Focus

Case 3a: $\alpha > 0$

The origin is called **Center**

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= -3x + 10y \\ \dot{y} &= -x + 3y \end{aligned}$$

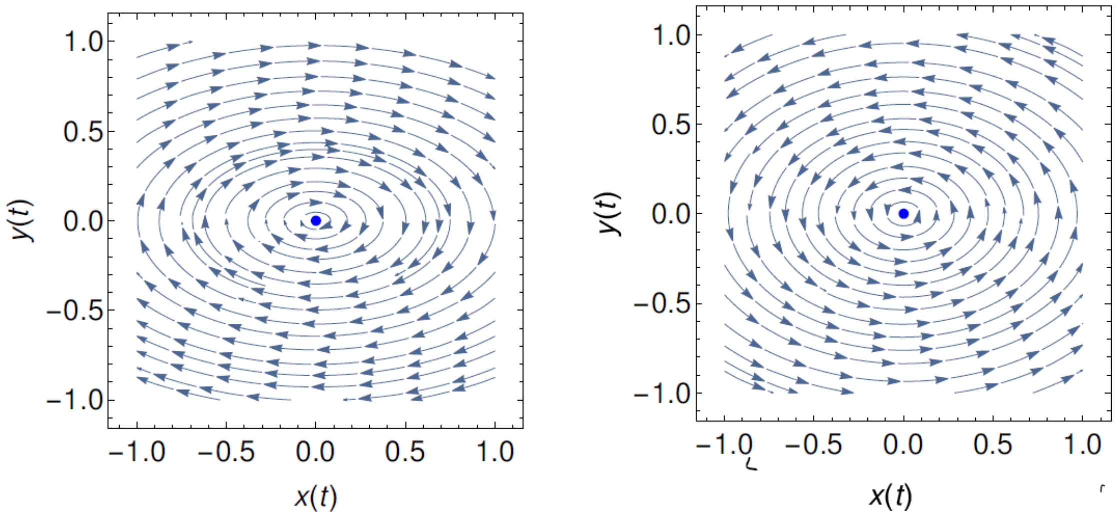


Figure 2.14: Center

- eigenvalues : $\lambda_1 = 0 + j$; $\lambda_2 = 0 - j \Longrightarrow$ *Center*
- eigen-vectors : $\vec{v_1} = \begin{pmatrix} 10 \\ 3+j \end{pmatrix}$, $\quad \vec{v_2} = \begin{pmatrix} 10 \\ 3-j \end{pmatrix}$

Phase Diagram

All of these behaviors can be classified according to the trace T_r and the determinant Det of the matrix A. Recall that for a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Find the eigenvalues of A :

$$det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - T_r\lambda + Det = 0$$

Thus the eigenvalues are

$$\begin{array}{lcl} Tr(A) & \equiv & a + d = \lambda_1 + \lambda_2 \Rightarrow \frac{1}{2}T_r(A) \quad = \quad m \quad \text{(mean)} \\ Det(A) & \equiv & ad - bc = \lambda_1\lambda_2 \qquad\qquad\quad = \quad p \ (product) \\ \lambda_1, \lambda_2 & = & m \pm \sqrt{m^2 - p} \end{array}$$

The values of (m, p) determine the equilibrium type.

- If $p < 0$, then the eigenvalues are real with opposite signs (**saddle node**).

- if $m^2 < p$, then the eigenvalues are complex with a real part (**spiral**: unstable if $m > 0$ and stable if $m < 0$).

- If $m = 0$ and $p > 0$, then the eigenvalues are purely imaginary (a **center**).

- $p > 0$ and $m^2 > p$ then the eigenvalues are real with the same sign (a **node**: stable if $m > 0$ and unstable if $m < 0$).

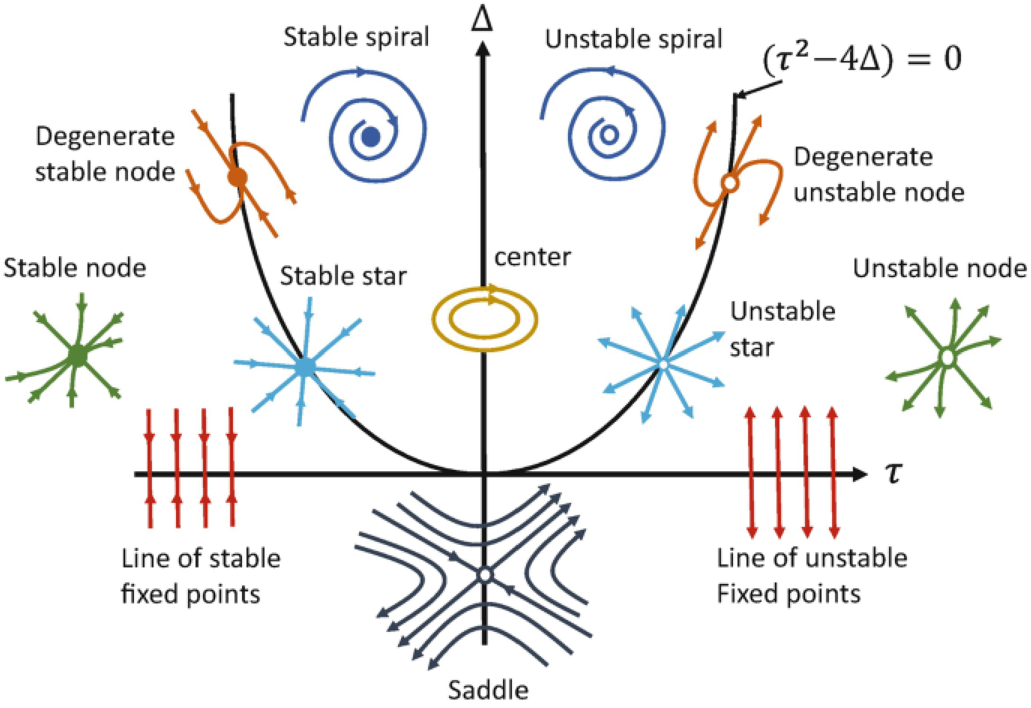


Figure 2.15: Phase plan analysis

For linear system
The global qualitative behavior is determined by the type of equilibium point.

For nonlinear system Only local qualitative behavior in the vicinty of equilibrium point is determined by the type of equilibium point.

2.1.4 Qualitative Behavior Near Equilibria

Given the nonlinear system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \tag{2.4}$$

Let us assume $x_e = (x_{1e}, x_{2e})$ is an equilibrium point of (2.4) i,e.,

$$f_1(x_{1e}, x_{2e}) = f_2(x_{1e}, x_{2e}) = 0$$

f_1, f_2 are continuously differentiable about (x_{1e}, x_{2e}).
Since we are interested in trajectories near (x_{1e}, x_{2e}), define

$$x_1 = x_{1e} + \tilde{x}_1, \qquad x_2 = x_{2e} + \tilde{x}_2$$

$\tilde{x}_1$, $\tilde{x}_2$ are small perturbations form equilibrium point.
Expanding (2.4) into its Taylor series

$$\dot{x}_1 = \dot{x}_{1e} + \dot{\tilde{x}}_1 = \underbrace{f_1(x_{1e}, x_{2e})}_{0} + \frac{\delta f_1(x)}{\delta x_1}\Big|_{x_e} \tilde{x}_1 + \frac{\delta f_1(x)}{\delta x_2}\Big|_{x_e} \tilde{x}_2 + H.O.T$$
$$\dot{x}_1 = \dot{x}_{1e} + \dot{\tilde{x}}_1 = \underbrace{f_1(x_{1e}, x_{2e})}_{0} + \frac{\delta f_1(x)}{\delta x_1}\Big|_{x_e} \tilde{x}_1 + \frac{\delta f_1(x)}{\delta x_2}\Big|_{x_e} \tilde{x}_2 + H.O.T$$

For sufficiently small neighborhood of equilibrium points, H.O.T. are negligible

$$\begin{array}{rcl} \dot{\tilde{x}}_1 & = & a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 \\ \dot{\tilde{x}}_2 & = & a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 \end{array}, \quad a_{i,j} = \frac{\delta f_i}{\delta x}\Big|_{x_e}, \quad i = 1, 2.$$

The equilibrium point of the linear system is

$$\tilde{x} = A\tilde{x}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} \end{bmatrix} = \frac{\delta f}{\delta x}\Big|_{x_e}$$

Matrix $\frac{\delta f}{\delta x}$ is called Jacobian Matrix .

The trajectories of the nonlinear system in a small neighborhood of an equilibrium point are close to the trajectories of its linearization about that point:

> if the origin of the linearized state equation is a stable (unstable) node, or a stable (unstable) focus or a saddle point, then in a small neighborhood of the equilibrium point, the trajectory of the nonlinear system will behave like a stable (unstable) node, or a stable (unstable) focus or a saddle point.

Example

$$\begin{array}{rcl} \dot{x}_1 & = & 3x_1 - x_1x_2 \\ \dot{x}_2 & = & -4x_2 + x_1x_2 \end{array}$$

- Equilibrium points: $f(\bar{x}) = 0 \Longrightarrow \bar{x} = (0, 0)$; $\bar{x} = (4, 3)$
- Linearization :

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 3 - x_2 & -x_1 \\ x_2 & -4 + x_1 \end{bmatrix}$$

- Linearization around $\bar{x} = (0, 0)$

$$\frac{\partial f(x)}{\partial x}\Big|_{\bar{x}=(0,0)} = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}; \quad \text{Eigenvalues} := \{3, -4\}$$

  ⇒ **Saddle** type of equilibrium.

- Linearization around $\bar{x} = (4, 3)$

$$\frac{\partial f(x)}{\partial x}\Big|_{\bar{x}=(4,3)} = \begin{bmatrix} 0 & -4 \\ 3 & 0 \end{bmatrix}; \quad \text{Eigenvalues} := \{0 \pm j\sqrt{12}\}$$

  ⇒ **Center** type of equilibrium.

Example

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_1 - x_1^3 - 0.2x_2\end{aligned}$$

- Equilibrium points: $f(\bar{x}) = 0 \Longrightarrow \bar{x} = (0,0)$; $\bar{x} = (1,0)$; $\bar{x} = (-1,0)$
- Linearization :

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 \\ 1-3x_1^2 & -0.2 \end{bmatrix}$$

- Linearization around $\bar{x} = (0,0)$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\bar{x}=(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix}; \quad \text{Eigenvalues} := \{-1.1, 0.9\}$$

$\Rightarrow$ **Saddle** type of equilibrium.

- Linearization around $\bar{x} = (1,0)$ and $\bar{x} = (-1,0)$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\bar{x}=(\pm1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -0.2 \end{bmatrix}; \quad \text{Eigenvalues} := \{-0.1 \pm \sqrt{2}\}$$

$\Rightarrow$ **Spiral Sink** type of equilibrium.

Example: ambiguous borderline case

$$\begin{cases} \dot{x}_1 = -x_2 + \underbrace{\mu x_1(x_1^2+x_2^2)}_{\text{nonlinear terms}} \\ \dot{x}_2 = x_1 + \overbrace{\mu x_2(x_1^2+x_2^2)} \end{cases} \tag{2.5}$$

Fixed point : $(x_{1e}, x_{2e}) = (0,0)$.
Linearization :

$$J = \left.\begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{pmatrix}\right|_{0,0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- It is important to note that the linearized system does not depend on the control parameter μ.

Classification of the fixed point of the linearized system.

- Trace of the system matrix is $T_r = 0$.
- Determinant of the system matrix is $p = 1$.

The linear fixed point is a **centre**.

Example : The Lotka-Volterra competitive cohabitation model from ecology competitive cohabitation of rabbits and sheep. The model has the following form:

$$\begin{cases} \dot{x} = x(3-x) - 2xy \\ \dot{y} = y(2-y) - xy \end{cases} \tag{2.6}$$

where x and y are the sizes of rabbit and sheep populations, respectively.

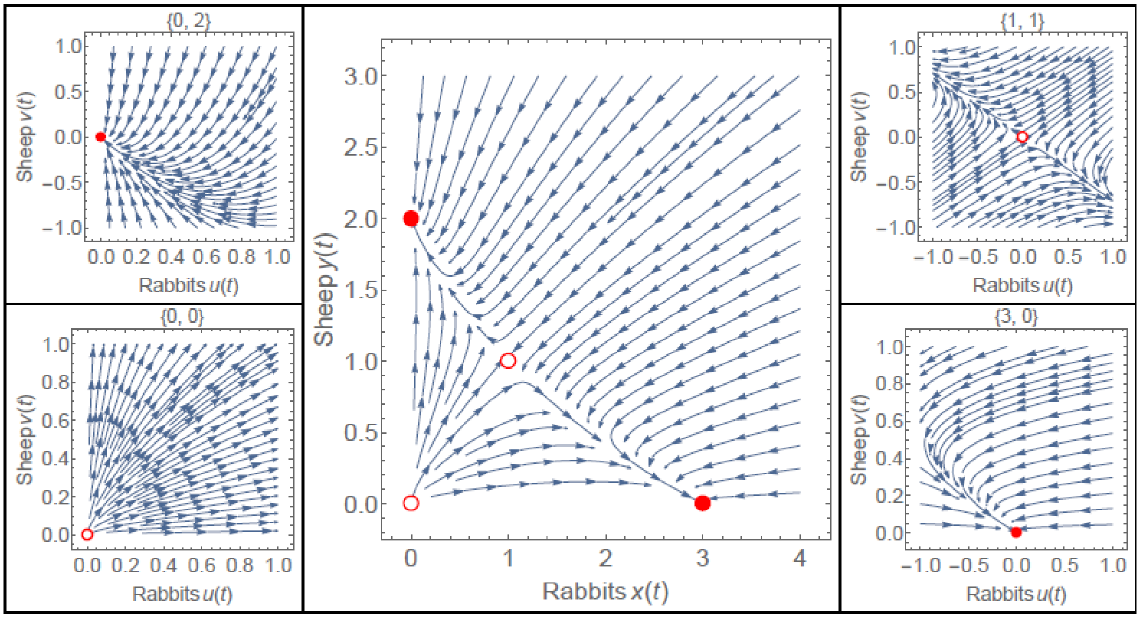


Figure 2.16: Lotka-Volterra competitive cohabitation model f

2.1.5 Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

Hyperbolic equilibrium point: linearization has no eigenvalues on the imaginary axis.

For hyperbolic equilibrium points, phase portraits of nonlinear systems near hyperbolic equilibria are qualitatively similar to the phase portraits of their linearization.

Hartman-Grobman Theorem:

If x_e is a hyperbolic equilibrium of a planar dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ then there is neighborhood U around x_e and a *homeomorphism*[1]

$$h : U \to \mathbb{R}^2$$

that maps the nonlinear trajectories in U to the linear trajectories in $\mathbb{R}^2$.

---

homeomorphism: a continuous map with a continuous inverse (i.e. a change of coordinates)

The change of coordinates is **unique** for all state trajectories until they stay in U.

Example: Consider the non-linear autonomous system

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= x_2 + x_1^2\end{aligned}$$

Equilibrium point : $\dot{x} = 0 \Rightarrow \bar{x} = (0,0)^T$.
Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 1 \Longrightarrow$ **saddle** type of equilibrium.

Example

$$\begin{aligned}\dot{x}_1 &= -x_2 + \mu x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= x_1 + \mu x_2(x_1^2 + x_2^2)\end{aligned} \tag{2.7}$$

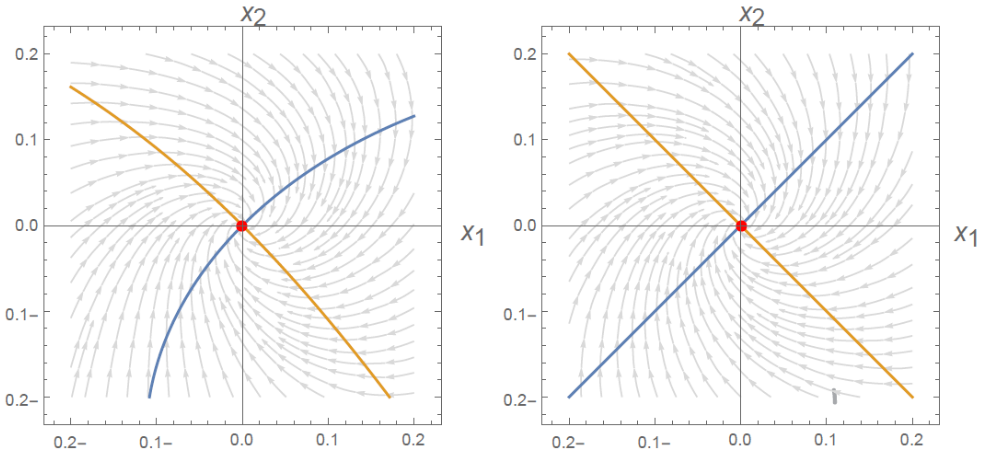


Figure 2.17: Qualitative Behavior Near Equilibria

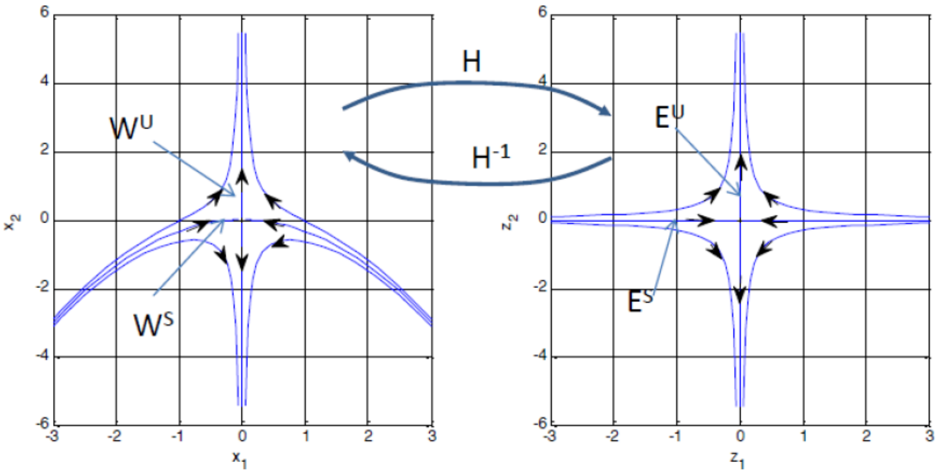


Figure 2.18: Hartman-Grobman Theorem

- There is only one equilibium point at (0,0), and the linearized system at this point is

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Longrightarrow \lambda_{1,2} = \{\pm J\} \tag{2.8}$$

  $\Longrightarrow$ the equilibrium point is **center**. Since this equilibrium point is *non-hyperbolic* $\Longrightarrow$ No conclusion about the behavior of the nonlinear system near (0,0)

System (2.7) is analysed in polar coordinates. Usually a direct coordinate transform

$$\left\{ \begin{array}{lcl} x_1 & = & r\cos\theta \\ x_2 & = & r\sin\theta \end{array} \right. , \left\{ \begin{array}{lcl} r & = & \sqrt{x_1^2 + x_2^2} \\ \theta & = & \tan\left(x_2/x_1\right) \end{array} \right. \tag{2.9}$$

where $r = r(t)$ and $\theta = \theta(t)$, is used. We are searching a system in the form:

$$\left\{ \begin{array}{lcl} \dot{r} & = & f_1(r,\theta) \\ \dot{\theta} & = & f_2(r,\theta) \end{array} \right. \tag{2.10}$$

where functions $f_1(r,\theta)$ and $f_2(r,\theta)$ are to be determined.

$$\begin{cases} \dot{x}_1 &= r\dot{r}\cos\theta - r\dot{\theta}\sin\theta = -r\sin\theta + \mu r^3\cos\theta \\ \dot{x}_2 &= r\dot{r}\sin\theta + r\dot{\theta}\cos\theta = r\cos\theta + \mu r^3\sin\theta \end{cases} \tag{2.11}$$

We are interested in temporal dynamics of (2.9)

$$\begin{cases} 2r\dot{r} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 & \Rightarrow \dot{r} = \dfrac{x_1f_1(x_1,x_2)+x_2f_2(x_1,x_2)}{r} \\ \sec^2\theta\dot{\theta} &= (1+\tan^2\theta)\dot{\theta} = \dfrac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2} & \Rightarrow \dot{\theta} = \dfrac{x_1f_2(x_1,x_2)-x_2f_1(x_1,x_2)}{x_1^2+x_2^2} \end{cases} \tag{2.12}$$

After developing (2.12). The system (2.7) has been represented in polar coordinates. Resulting decoupled equations

$$\begin{array}{rcrcl} \dot{x}_1 &=& -x_2 &+& \mu x_1(x_1^2+x_2^2) \\ \dot{x}_2 &=& x_1 &+& \mu x_2(x_1^2+x_2^2) \end{array} \Longrightarrow \begin{cases} \dot{r} &= \mu r^3 \\ \dot{\theta} &= 1 \end{cases}$$

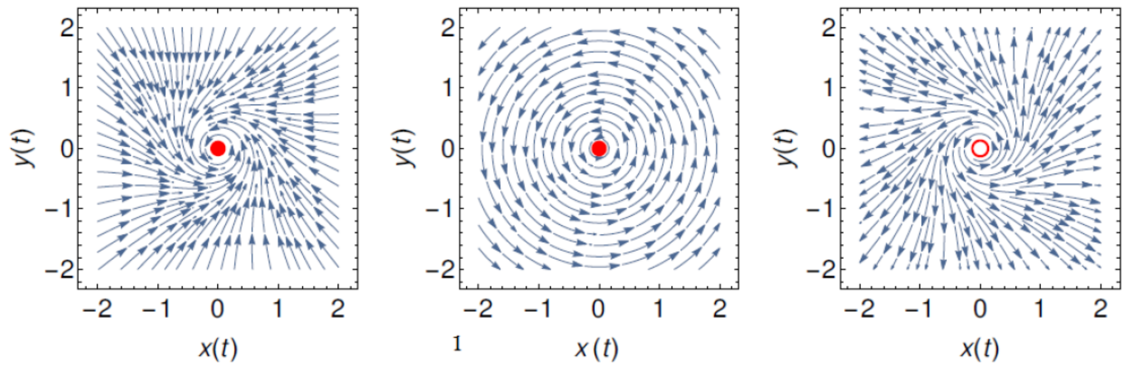


Figure: $\mu < 0,$ $\mu = 0,$ $\mu > 0$

Figure 2.19: Example 1: Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

2.1.6 Non-existence of Periodic Orbits

Bendixson criterion gives a sufficient condition for detecting the absence of periodic orbits for second-order systems (Limit cycles or neutrally stable cycles).

Bendixson criterion:

For a time-invariant planar system

$$\dot{x}_1 = f_1(x_1,x_2), \quad \dot{x}_2 = f_2(x_1,x_2)$$

If $div(f) = \nabla.f(x) = [\partial/\partial x_1 \quad \partial/\partial x_2]\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \partial f_1/\partial x_1 + \partial f_2/\partial x_2$ is not identically zero and does not change sign in a simply connected region D, then there are no periodic orbits lying entirely in D.

Example 1 : $\dot{x} = Ax, x \in \mathbb{R}^2$ can have periodic orbits only if div f= trace(A)=0.,

$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}$$

$trace(A) = 0 \Longrightarrow$ there are periodic orbits.

Example 2:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2, \qquad \delta > 0 \end{aligned}$$

$$\nabla.f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 - \delta$$

$$\nabla.f(x) = 0, \quad \text{then} \quad x_1 = \pm\sqrt{\delta}$$

Therefore, no periodic orbit can lie entirely in the region

$$x_1 \in]-\infty, -\sqrt{\delta}[, \quad]-\sqrt{\delta}, \sqrt{\delta}[, \quad]\sqrt{\delta}, +\infty[$$

$$\begin{cases} x_1 & \leq & -\sqrt{\delta} & \text{where} & \nabla.f(x) \geq 0. \\ |x_1| & \leq & \sqrt{\delta} & \text{where} & \nabla.f(x) \leq 0. \\ x_1 & \geq & \sqrt{\delta} & \text{where} & \nabla.f(x) \geq 0. \end{cases}$$

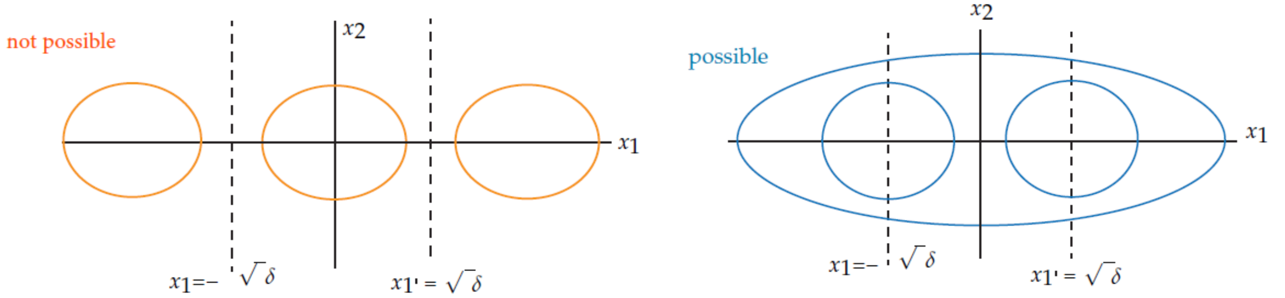


Figure 2.20: Example 2

2.1.7 Periodic Orbits in the Plane

Let $\phi(t, x_0)$ denotes the solution of $\dot{x} = f(x)$ with initial condition $x(0) = x_0$.

Definition: Invariant sets

A set $M \subseteq \mathbb{R}^n$ is **positively invariant** if, for each $x_0 \in M$, $\phi(t, x_0) \in M$ for all $t \geq 0$.

Theorem

If $V : \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^1$ and $M = \{x : V(x) \leq c\}$, then M is invariant if

$$f(x).\nabla V(x) \leq 0 \quad \forall x : V(x) = c$$

i.e. if x is on the boundary of M, then the vector $f(x)$ points into M.

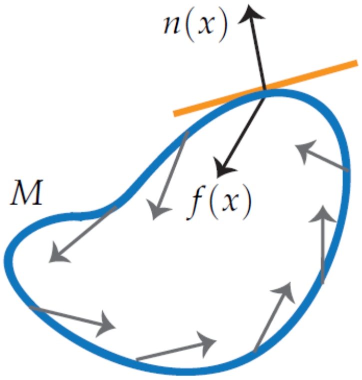
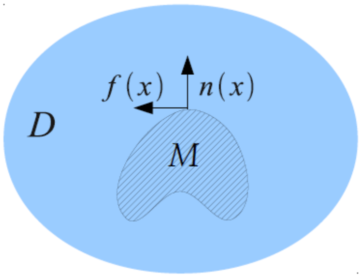


Figure 2.21: nvariant set

If $f(x)^T.\vec{n} \leq 0$ then M is positively invariant.
$\vec{n}$: outward normal on a boundary of M.
Along boundary of M:for all $x \in \partial M \Rightarrow [f(x)]^T.\vec{n} \leq 0$.

Example: Consider a closed orbit $\dot{x} = f(x)$.
$f(x)$ is tangential to the trajectory x. $\vec{n}$: outward normal on a boundary of M. Along this closed trajectory : $f^T(x).\vec{n} = 0$. The interior of any closed trajectory is a positively invariant set.



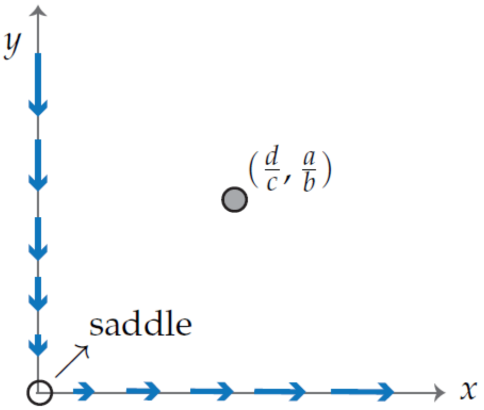
Example: Predator-prey model

$$\begin{aligned} \text{prey :} \quad & \dot{x}_1 &=& \ (a - bx_2)x_1 \\ \text{predator :} \quad & \dot{x}_1 &=& \ (cx_1 - d)x_2 \end{aligned}$$

a, b, c, d positive parameters.
Equilibrium points : $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} d/c \\ a/b \end{bmatrix}.$
Clearly $[f(x)]^T.\vec{n} = 0$ along the boundary of $M = \{x_1 \geq 0, x_2 \geq 0\}$, which means the first quadrant M is positively invariant.
Linearization around $\bar{x} = (0,0)$: $A = \frac{\partial f}{\partial x}\Big|_{\bar{x}} = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}.$

$\lambda_1 = a > 0$ and $\lambda_2 = -d < 0 \Longrightarrow$ **saddle** type of equilibrium.
Example :

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{aligned}$$



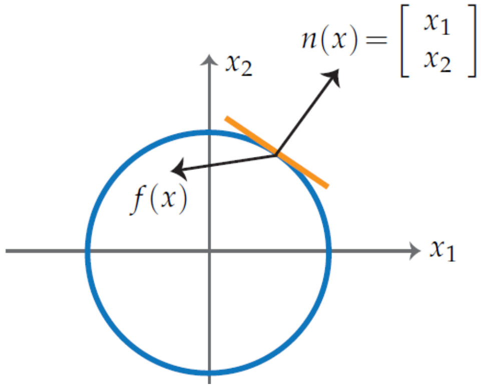
Show that $B_r := \{x \in \mathbb{R}^2 / x_1^2 + x_2 \leq r^2\}$ is positively invariant for sufficiently large r (to be determined). We want to calculate $[f(x)]^T n(x)$

$$V(x) = x_1^2 + x_2^2 = r^2 \Rightarrow n(x) = \nabla V(x) = \begin{bmatrix} \partial V/\partial x_1 \\ \partial V/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\begin{aligned} [f(x)]^T.n(x) &= = f_1 \tfrac{\partial V}{\partial x_1} + f_2 \tfrac{\partial V}{\partial x_2} \\ &= 2x_1(x_1 + x_2 - x_1(x_1^2 + x_2^2)) + 2x_2(-2x_1 + x_2 - x_2(x_2^2 + x_2^2)) \\ &= -2(x_1^2 + x_2^2)^2 + 2x_1^2 + 2x_2^2 - 2x_1x_2 \end{aligned}$$

$$-2x_2x_2 \leq x_1^2 + x_2^2 \qquad \text{(completion of squares).}$$

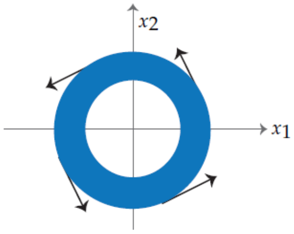
Therefore $[f(x)]^T.n(x) \leq -2r^2(r^2 - 3/2)) \leq 0$ if $r^2 \geq 3/2$.



2.1.8 Existence Theorem of Limit Cycle

Poincaré-Bendixson Theorem:

Let M be a compact (closed and bounded) set in $\mathbb{R}^2$, which is positively invariant for $\dot{x} = f(x)$, $x \in \mathbb{R}^2$. If M does not contain an equilibrium point, then it contains a periodic orbit.



The "no equilibrium condition" in PB Theorem can be relaxed as: " M can have one equilibrium point which is either an unstable focus or an unstable node, then there is a periodic orbit.

Example: harmonic oscillator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For any $R > r > 0$, the ring $\{x : r^2 \leq x_2 + x_2 \leq R^2\}$ is compact, invariant and contain no equilibria. $[f(x)]^T.n(x) = 0$ everywhere and PB Theorem state there exists a periodic orbit (or more) in M.

Example:2

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 x_2^2 \\ \dot{x}_1 &= -x_1 + x_1^2 x_2 \end{aligned}$$

Linearization around the equilibrium at $\bar{x} = (0\ 0)^T$ yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which exhibits a continuum of periodic solutions. However, for this nonlinear system, we have

$$\nabla.f(x) = x_1^2 + x_2^2 > 0, \quad \forall x \neq 0$$

Hence, Bendixson theorem leads to the conclusion that this dynamical system has no nontrivial periodic solutions.

Example:3

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{aligned}$$

$[f(x)]^T.n(x) \leq 0$ iff $r^2 = x_1^2 + x_2^2 > 3/2$. i.e. $B_r = x \in \mathbb{R}^2 / x_1^2 + x_2^2 \leq r^2$ is positively invariant $r \geq \sqrt{3/2}$ but contains the equilibrium $x_e = 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{x_e=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, \ \lambda_{1,2} = 1 \pm j\sqrt{2}, \text{ unstable focus}$$

Therefore, B_r must contain a periodic orbit.

2.1.9 Limit Cycle

Stable Limit Cycle

All trajectories in the vicinity of the limit cycle converges to it as $t \rightarrow \infty$

Example:

$$\begin{array}{rcl} \dot{x}_1 &=& x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_1 &=& -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{array}$$

$$\Longrightarrow \left\{ \begin{array}{rcl} \dot{r} &=& -r(r^2 - 1) \\ \dot{\theta} &=& -1 \end{array} \right. \tag{2.13}$$

- if $r > 1 \rightarrow \dot{r} > 0$ converging
- if $r > 1 \rightarrow \dot{r} < 0$ converging
- if $r = 1 \rightarrow \dot{r} = 0$ remaining

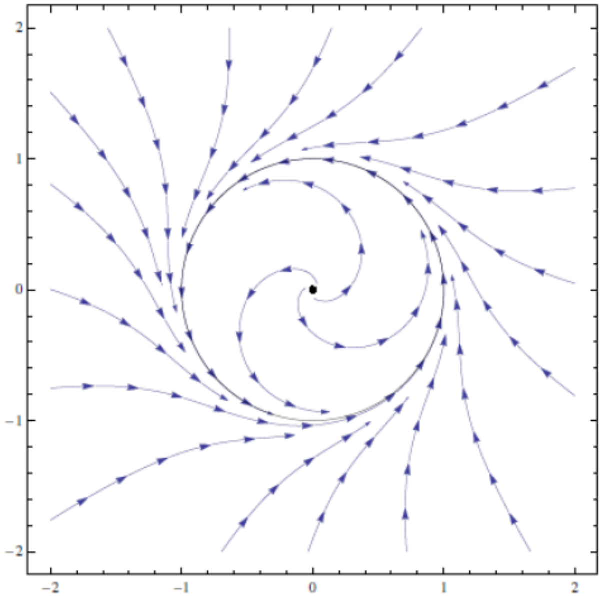


Figure 2.22: Stable Limit Cycle

Unstable Limit Cycle

All trajectories in the vicinity of the limit cycle diverges from it as $t \rightarrow$

Example

$$\begin{array}{rcl} \dot{x}_1 &=& x_2 + x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_1 &=& -x_1 + x_2(x_1^2 + x_2^2 - 1) \end{array}$$

$$\Longrightarrow \left\{ \begin{array}{rcl} \dot{r} &=& r(r^2 - 1) \\ \dot{\theta} &=& -1 \end{array} \right. \tag{2.14}$$

- if $r < 1 \to \dot{r} < 0$ diverging
- if $r > 1 \to \dot{r} > 0$ diverging
- if $r = 1 \to \dot{r} = 0$ remaining

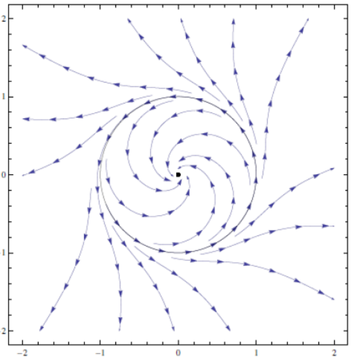


Figure 2.23: Unstable Limit Cycle

half-stable Limit Cycle

Some of the trajectories in the vicinity of the limit cycle converges to it, while others diverge from it as $t \to \infty$

Example:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_1 & = & -x_1 - x_2(x_1^2 + x_2^2 - 1)^2 \end{array}$$
$$\Longrightarrow \left\{ \begin{array}{rcl} \dot{r} & = & -r(r^2-1)^2 \\ \dot{\theta} & = & -1 \end{array} \right. \tag{2.15}$$

- if $r < 1 \to \dot{r} < 0$ diverging
- if $r > 1 \to \dot{r} < 0$ converging
- if $r = 1 \to \dot{r} = 0$ remaining

2.1.10 Bifurcation: Holf Bifurcation

Example: Supercritical Hopf bifurcation

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_2 + x_1(\mu - x_1^2 - x_2^2) \\ \dot{x}_2 & = & +x_1 - x_2(\mu - x_1^2 - x_2^2) \end{array} \right. \Longrightarrow \left\{ \begin{array}{rcl} \dot{r} & = & r(\mu - r^2) \\ \dot{\theta} & = & 1 \end{array} \right. \tag{2.16}$$

Equilibrium points : $r(\mu - \bar{r}^2) = 0$.
Note that a positive equilibrium for the r subsystem means a limit cycle in the $(x1, x2)$ plane.

- $\mu < 0$: stable equilibrium at $r = 0$.

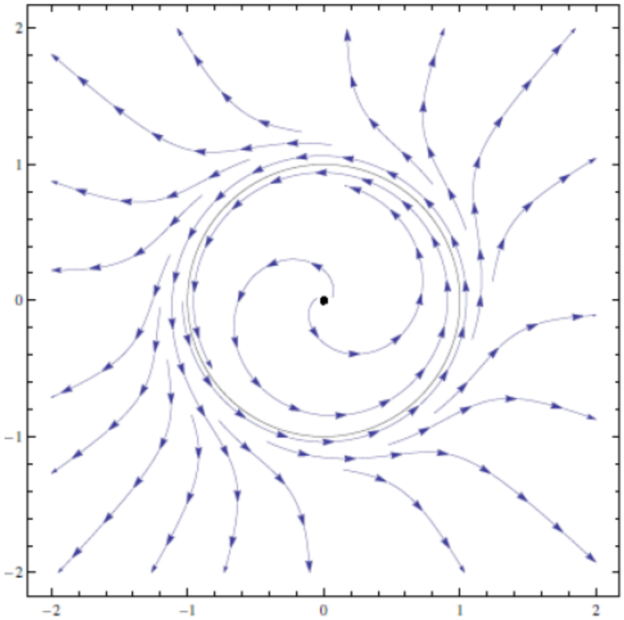


Figure 2.24: Unstable Limit Cycle

- $\mu > 0$ unstable equilibrium point ar $r = 0$ and stable limit cycle at $r = \sqrt{\mu}$.
- The origin loses stability at $\mu = 0$ and a stable limit cycle emerges.

In Supercritical Hopf bifurcation by increase of μ near zero, the stable equilibrium point becomes unstable but a stable limit cycle appears. Hence, this is a Safe bifurcation.

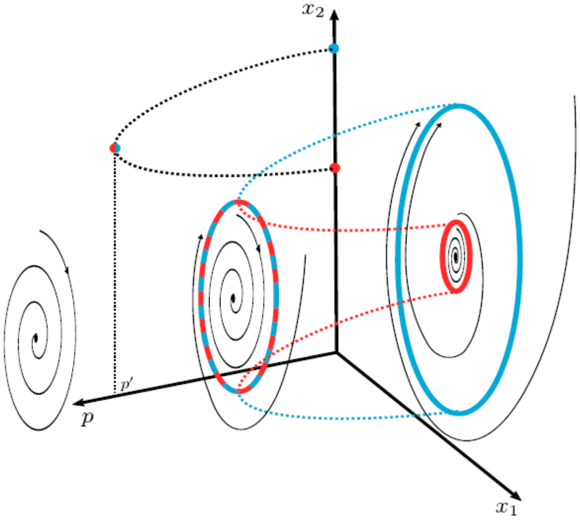


Figure 2.25: Bifurcation: Holf Bifurcation

Chapter 3

Lyapunov Stability Theory

3.1 INTRODUCTION

Stability theory is central to understanding the behavior of dynamical systems and is divided into three key areas:

- Stability of equilibrium points, focusing on whether a system returns to its steady state after perturbations.
- Stability of periodic orbits, analyzing the behavior of oscillatory or cyclic solutions.
- Input/output stability, examining how systems respond to external inputs and disturbances.

These components provide a framework for studying the robustness and dynamics of systems under various conditions.

Alexander Mikhailovich Lyapunov (1957-1918)

- Russian mathematician and physicist.
- Known for his development of the stability theory of dynamical systems.

> If the total energy is dissipated, then the system must be stable.

3.2 Stability of Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{3.1}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a *locally Lipschitz* map from a domain $\mathcal{D} \subset$ into $\mathbb{R}^n$.

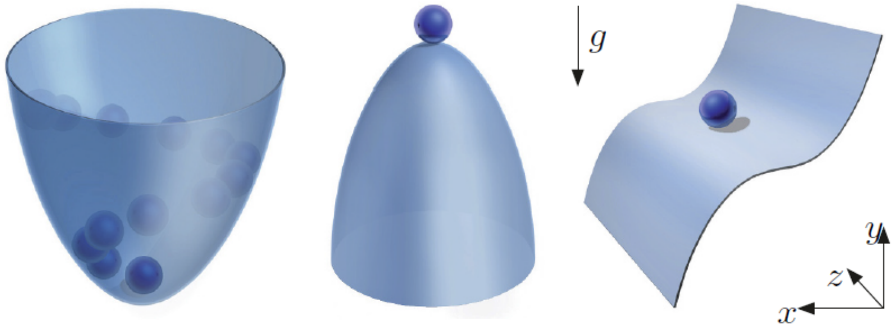
Suppose that the system (3.1) has an equilibrium point $\bar{x} \in \mathcal{D}$, i.e., $f(\bar{x}) = 0$.

If $\bar{x}$ is an equilibrium state, set $z = x - \bar{x}$ and, from (3.1), obtain

$$\dot{z} = f(z + \bar{x}) \tag{3.2}$$

- $z \bar{=} 0$ is an equilibrium state for (3.2).

- $x(t) = \phi(t, x_0)$ is a state trajectory for (3.1) $\Longleftrightarrow$ $z(t) = x(t) - \bar{x}$ is a state trajectory of (3.2) with $z(0) = x_0 - \bar{x}$



3.2.1 Classification of stability

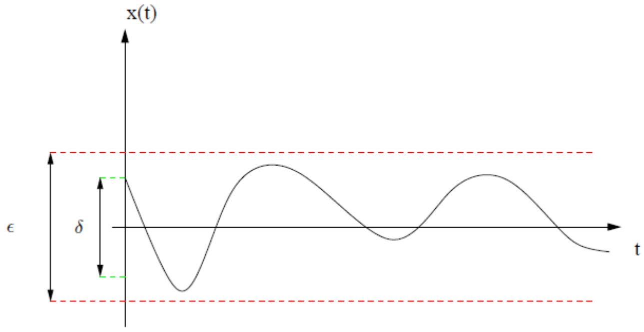
Definition (Lyapunov stability)

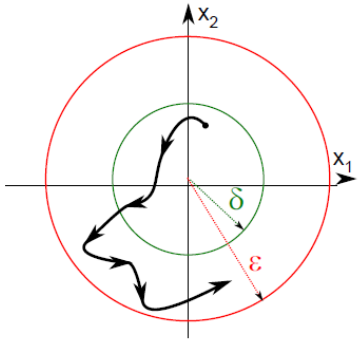
The equilibrium point $\bar{x} = 0$ of $\dot{x} = f(x)$ is

- **stable**, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \Longrightarrow \|x(t)\| < \epsilon, \forall t \geq 0.$$

- **unstable**, if it is not stable.





Note :Stability is a property of the equilibrium, not of the system.

Stability of the equilibrium is equivalent to stability of the system only when there exists only one equilibrium (e.g., linear systems). In this case **stability** $\equiv$ **global stability.**

The region of **attraction** of the equilibrium point $\bar{x} = 0$ of (3.1) is is the set of all initial conditions $x(0)$ for which

$$x(t) \to 0 \quad \text{as} \quad t \to \infty.$$

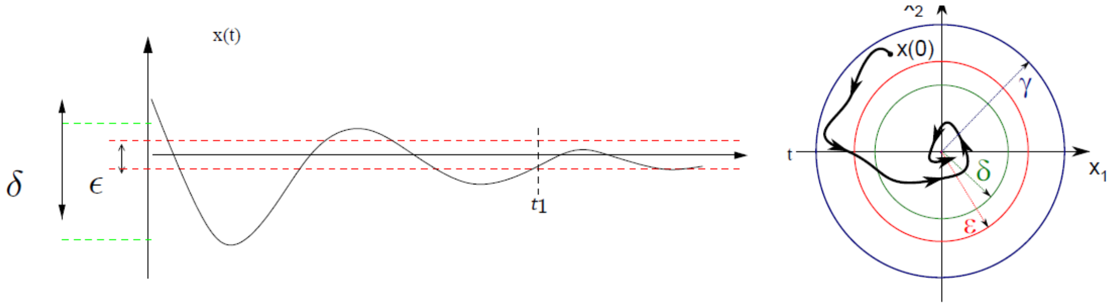
Asymptotic Stability

The equilibrium point $\bar{x} = 0$ of (3.1) is

- **attractive**, if there exist δ such that :

$$\|x(0)\| < \delta \implies \lim_{t\to\infty} x(t) = 0.$$

- **local asymptotically stable** (a.s) if it is **stable** and **attractive**



Exponential stability

The equilibrium point $\bar{x} = 0$ of (3.1) is

- **exponentially stable** (e.s), if there exists $\alpha >, \beta > 0$ and $\delta > 0$ such that

$$\|x(0)\| < \delta \implies \|x(t)\| \leq \beta \|x(0)\| e^{-\alpha t}, \forall t \geq 0$$

exponential stability is a special case of asymptotic stability.

ES ⇒ AS ⇒ Stability. All the opposite implications are false.

Global stability

The equilibrium point $\bar{x} = 0$ of (3.1) is

- **globally asymptotically stable** (g.a.s) if it is stable and globally attractive, i.e $\lim_{t\to\infty} x(t) = 0$. for all $x(0) \in \mathbb{R}^n$.
- **globally exponentially stable**(g.e.s), for all $x(0) \in \mathbb{R}^n$, there exists $\beta > 0$ and $\alpha > 0$ such that

$$\|x(t)\| \leq \beta \|x(0)\| e^{-\alpha t}, \forall t \geq 0$$

3.2.2 Remarks on stability

Stability and attractivity are two different notions

- Stability looks at whether the trajectories remain in some neighbourhood of the equilibrium.
- Attractivity looks at whether the trajectories converge to the equilibrium.

Chaotic attractor: The butterfly system has a unique equilibrium point at 0, which is a globally attracting point but is unstable.

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_2^2 \\ \dot{x}_2 &= 2x_1x_2 \end{aligned}$$

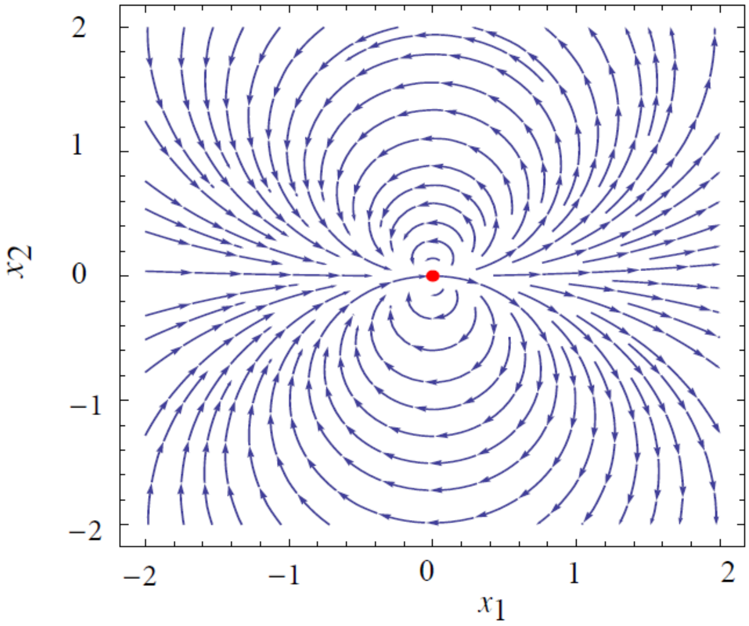


Figure 3.1: The butterfly system

There exist Lyapunov-stable sets that are not attractors.

Example: stable system but not attractor

$$\begin{array}{rcl}\dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -\sin(x_1)\end{array}$$

Show that an equilibrium point $\bar{x} = [2k\pi, 0]^T$ is stable but not attractor.

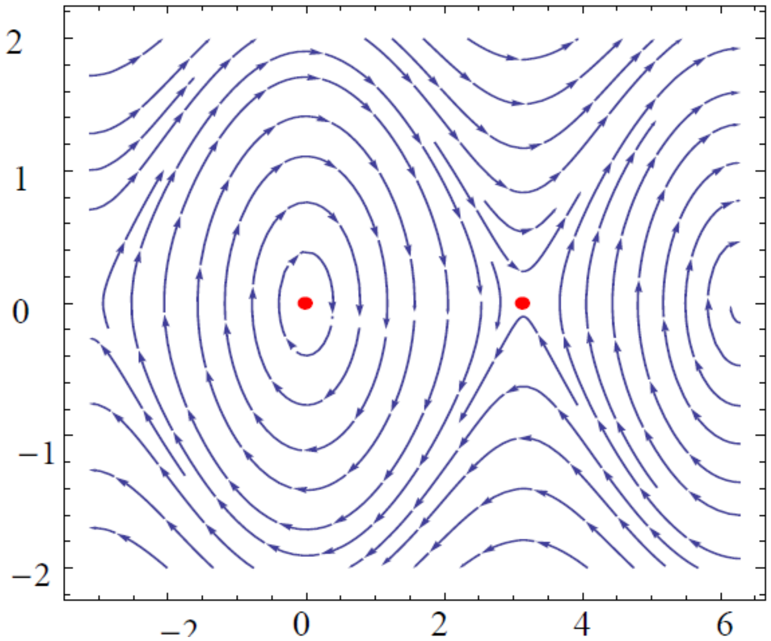


Figure 3.2: Phase Portraits of a Simple Pendulum

Show that an equilibrium point $x_e = [2k\pi, 0]^T$ is stable but not .attractor

3.3 Lyapunov direct method

3.3.1 Positive Definite Functions

Definition: Positive Definite Functions

A function $V : \mathcal{D} \to \mathbb{R}$ is *positive semi definite* in $\mathcal{D}$ if

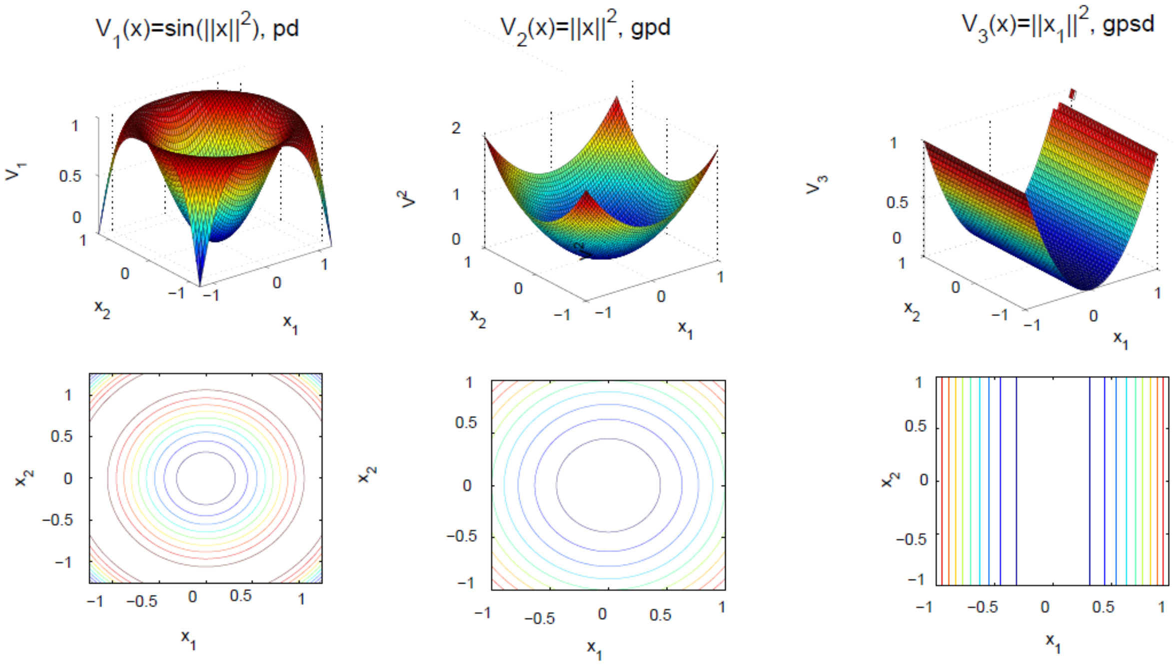
(i). $V(x) = 0$ if and only if $x = 0$.

(ii). $V(x) \geq 0$, $\forall x$in $\mathcal{D} - \{0\}$.

A function $V : \mathcal{D} \to \mathbb{R}$ is *positive definite* in $\mathcal{D}$ if

(ii'). $V(x) > 0$, $\forall x$in $\mathcal{D} - \{0\}$.

A function $V : \mathcal{D} \to \mathbb{R}$ is *negative definite* in $\mathcal{D}$ if $-V$ is positive definite (semi definite).



V_1(x) pd

V_2(x) gpd

V_3(x) gpsd

Figure 3.3: Positive Definite Functions

3.3.2 Derivative along the trajectory

Definition: Lyapunov Function

Let $V : \mathcal{D} \to \mathbb{R}$ be a *continuously differentiable* function defined in a domain $D \in \mathbb{R}^n$ that contains the origin. The derivative of V along the trajectory (solution) of $\dot{x} = f(x)$ denoted by $\dot{V}(x)$ is given by

$$\begin{aligned} \dot{V}(x) = \frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt} = \nabla V . f(x) \\ &= \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \end{aligned} \tag{3.3}$$

Equation (3.3) is called Lie derivative of V along f

- it measures how much V decreases along state trajectories.

Example : The system under consideration is a nonlinear second-order differential equation that describes the motion of a mass M subjected to both nonlinear damping and nonlinear elastic forces . The governing equation is expressed as:

$$M\ddot{x} = \underbrace{-b\dot{x}|\dot{x}|}_{\text{NL damping}} - \underbrace{\left(k_0 x + k_1 x^3\right)}_{\text{NL elasticforce}}$$

Equilibrium Point Analysis

- Defining $x_1 = x$ and $x_2 = \dot{x}_1$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{b}{M} x_2 |x_2| - \frac{k_0}{M} x_1 - \frac{k_1}{M} x_1^3 \end{cases}$$

  Thus, the equilibrium points are:

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Linearization and Jacobian Matrix

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{M} - 3\frac{k_1}{M} x_1^2 & -\frac{2b}{M} x_2 \mathrm{sgn}(x_2) \end{bmatrix} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{M} & 0 \end{bmatrix}$$

- Eigenvalue Analysis : the eigenvalues are:

$$\lambda_{1,2} = \pm j\sqrt{k_0/M}.$$

  which are purely imaginary.

The equilibrium point corresponds to a *center* in the linearized system, but its stability in the full nonlinear system remains inconclusive without further investigation. The presence of purely imaginary eigenvalues indicates that the linearized system does not provide conclusive information about the stability of the equilibrium point $\bar{x} = [0, 0]^T$. in the full nonlinear system. Specifically:

- The equilibrium point cannot be classified as asymptotically stable (AS) or exponentially stable (ES) based on the linearized system alone.
- Further analysis, such as Lyapunov's direct method or numerical simulations, is required to determine the stability properties of the nonlinear system.

Energy Analysis of the System

To analyze the stability of the system, we consider the total energy of the system:

$$V(x) = \underbrace{\frac{1}{2} M x_2^2}_{kinetic} + \underbrace{\frac{1}{2} k_0 x^2 + \frac{1}{4} k_1 x_1^4}_{potential}$$

The total energy $V(x)$ is a positive-definite function of the state variables x_1 and x_2. This means $V(x) > 0$ for all nonzero states ($x_1 \neq 0$ or $x_2 \neq 0$), and $V(x) = 0$ only at the equilibrium point $x_1 = 0$ and $x_2 = 0$.

Zero energy corresponds to the equilibrium state, i.e., $V(x) = 0$ iff $x_1 = 0$ and $x_2 = 0$.

This property makes $V(x)$ a candidate for a Lyapunov function , which can be used to study the stability of the system.

To understand how the total energy evolves over time, we compute the time derivative of $V(x)$ along the trajectories of the system. The instantaneous rate of change of $V(x)$ is given by:

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \cdot \begin{bmatrix} \frac{dx_1}{dt} & \frac{dx_2}{dt} \end{bmatrix}^T.$$

Substituting the expressions for $V(x)$ and the dynamics of the system:

$$\begin{aligned}\dot{V}(x) &= \left(k_0 x_1 + k_1 x_1^3\right)\dot{x}_1 + M x_2 \dot{x}_2 \\ &= \left(k_0 x_1 + k_1 x_1^3\right) x_2 + M x_2 \left(-\frac{b}{M} x_2 |x_2| - \frac{k_0}{M} x_1 - \frac{k_1}{M} x_1^3\right).\end{aligned}$$

Simplifying further:

$$\begin{aligned}\dot{V}(x) &= \left(k_0 x_1 + k_1 x_1^3\right) x_2 - b x_2^2 |x_2| - \left(k_0 x_1 + k_1 x_1^3\right) x_2 \\ &= -b x_2^2 |x_2|.\end{aligned}$$

Interpretation of $\dot{V}(x)$:

The term $-b x_2^2 |x_2|$ represents the rate of energy dissipation due to the nonlinear damping force $-b x_2 |x_2|$. Since $b > 0$ and $x_2^2 |x_2| \geq 0$, it follows that:

$$\dot{V}(x) = -b x_2^2 |x_2| \leq 0.$$

$-b x_2^2 |x_2| \leq 0$ independently of $x(0)$ ⇒ the energy can only decrease with time independently of $x(0)$

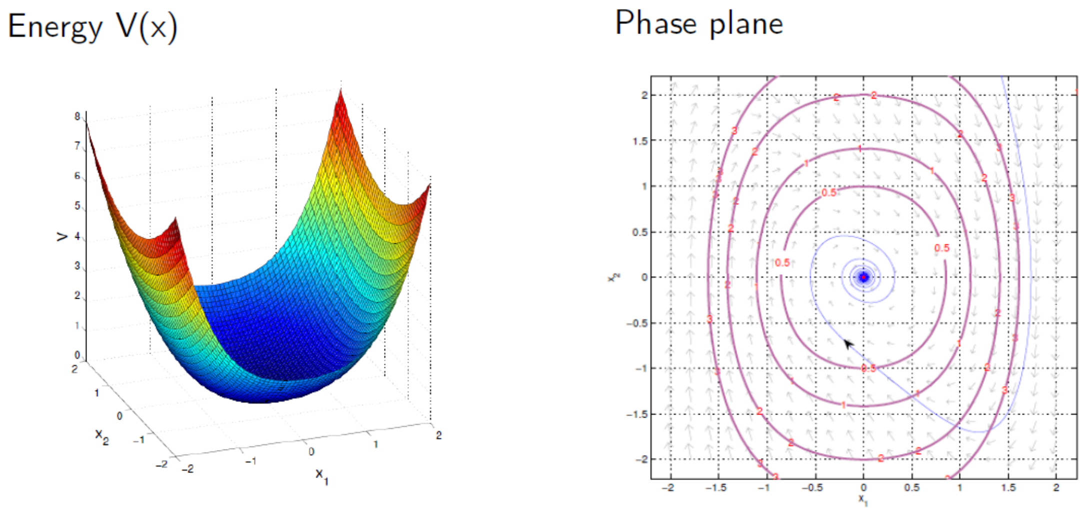


Figure 3.4: Energy Analysis of the System

- Energy is a "measure" of the distance of x from the origin if it can only decrease, then $\bar{x} = 0$ should be stable.

- Lyapunov direct method is based on energy-like functions V (x) and the analysis of the function.

3.3.3 Lyapunov Stability Theorem

Theorem

1. Let $\bar{x} = 0$ be an equilibrium for $\dot{x} = f(x)$ and $\mathcal{D} \in \mathbb{R}^n$ be a domain containing $\bar{x} = 0$. If there exists a *continuously differentiable* function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \forall x \in \mathcal{D} - \{0\} \quad \text{(positive definite)}$$

and

$$\dot{V}(x) := \nabla V^T(x) f(x) \leq 0 \quad \forall x \in \text{in } \mathcal{D} \quad \text{(negative semidefinite)}$$

then, $\bar{x} = 0$ is **stable**.

2. If $\dot{V}(x) < 0$, $\forall x \in \mathcal{D} - \{0\}$ (negative definite)
then, $\bar{x} = 0$ is **asymptotically stable**.

3. If, in addition, $\mathcal{D} = \mathbb{R}^n$, and

$$\|x\| \to \infty \Longrightarrow V(x) \to \infty. \quad \text{(radially unbounded)}$$

then $\bar{x} = 0$ is **globally asymptotically stable.**

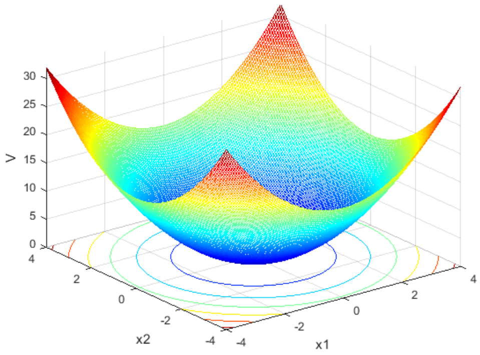
Example

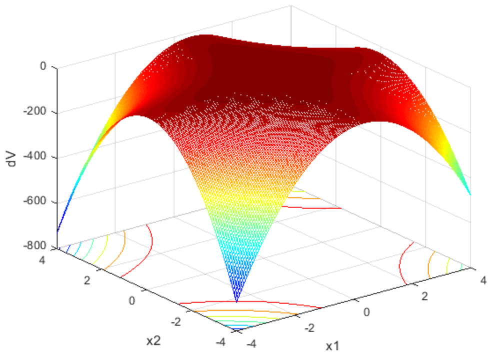
$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= x_1 - x_2 - x_1^2 x_2 \end{aligned}$$

Study the stability of the equilibrium state $\bar{x} = 0$.
Take the Lyapunov function $V(x) = x_1^2 + x_2^2$ (positive definite in $\mathbb{R}^2$)

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x) = 2x_1(-x_1 + x_2^2) + 2x_2(x_1 - x_2 - x_1^2 x_2) \\ &= -(x_1 - x_2)^2 - x_2^2(1 - x_1)^2 - x_1^2(1 + x_2^2) \end{aligned}$$

$\dot{V}(x)$ is negative definite $\Longrightarrow$ the system is stable.





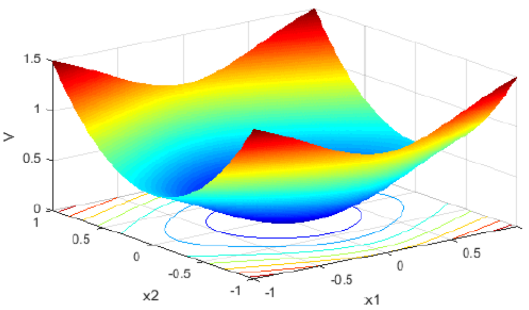
Q.why is radially unboundedness required for G.A.S ?

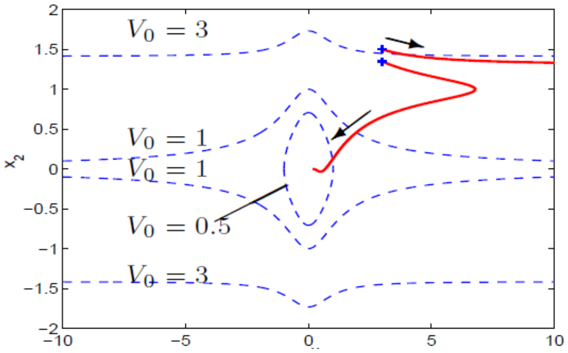
Example: Consider the following *positive definite function*

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

if $x = (x_1, 0)$ then $\|x\| \to \infty \quad as \quad x_1 \to \infty$.
But $V(x)$ will approach 26 (not radially unbounded)

$$\lim_{x_1\to\infty} V(x) = \lim_{x_1\to\infty} \frac{1+x_1^2}{x_1^2} + (5)^2 = 26 \neq +\infty$$





3.3.4 Conservation and Dissipation

Conservation of energy:

$\dot{V}(x) = \dfrac{\partial V}{\partial x} f(x) = 0$, i.e., the vector field $f(x)$ is everywhere orthogonal to the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$.

Example: Total energy of a lossless mechanical system or total fluid in a closed system.

Dissipation of energy

$\dot{V}(x) = \dfrac{\partial V}{\partial x} f(x) \leq 0$, i.e., the vector field $f(x)$ and the normal $\frac{\partial V}{\partial x}$ to the level surface $(V(x) = c)$ make an obtuse angle.

Example : Total energy of a mechanical system with damping or total fluid in a system that leaks.

Not necessary to compute state trajectories: it is enough to check the sign of V and $\dot{V}$ in a neighborhood of the origin.

Example:

$$\begin{aligned} \dot{x}_1 &= x_1\left(x_1^2 + x_2^2 - 2\right) - 4x_1x_2^2 \\ \dot{x}_2 &= x_2\left(x_1^2 + x_2^2 - 2\right) + 4x_1^2x_2 \end{aligned}$$

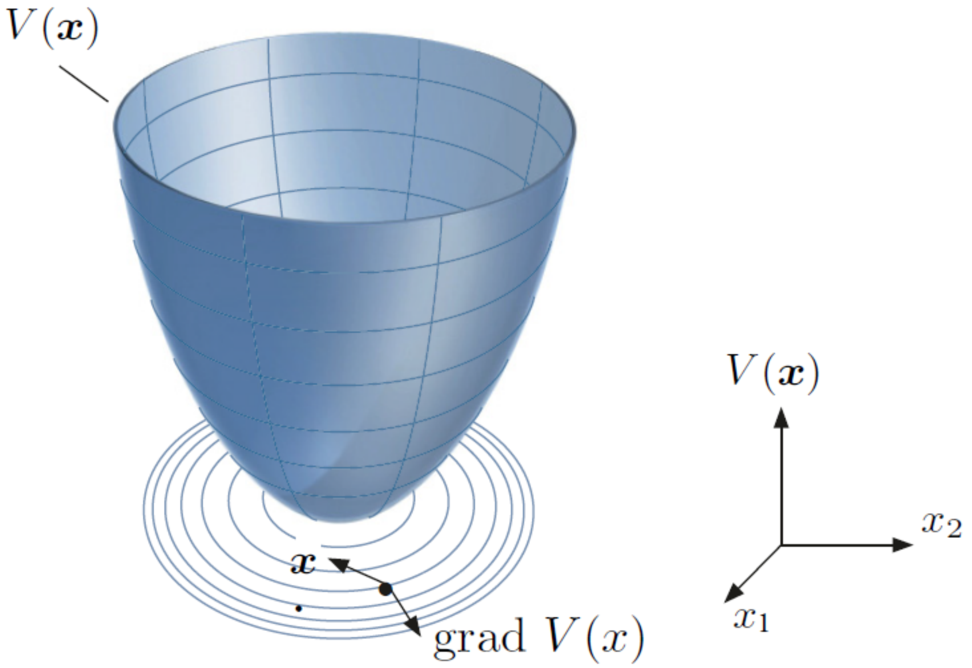
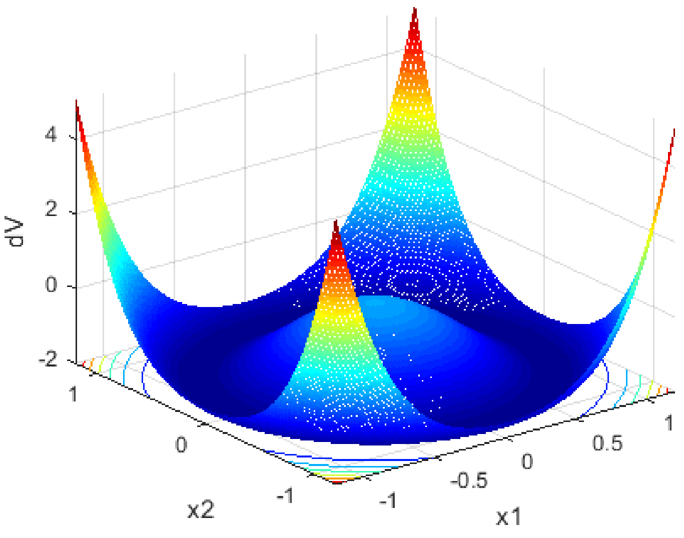


Figure 3.5: Illustration of the equation $\dot{V}(x) = \dot{x}^T \text{grad}(V(x)) < 0$

Study the stability of the equilibrium state $\bar{x} = 0$
Candidate Lyapunov function: $V(x) = x_1^2 + x_2^2$ (positive definite in $\mathbb{R}^2$)

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x) \\ &= 2x_1\left(x_1\left(x_1^2 + x_2^2 - 2\right) - 4x_1x_2^2\right) + 2x_2\left(x_2\left(x_1^2 + x_2^2 - 2\right) + 4x_1^2x_2\right) \\ &= 2\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - 2\right)\end{aligned}$$

In the set level $x_1^2 + x_2^2 - 2 < 0$ one has $\dot{V}$ is negative definite, therefore $\bar{x} = 0$ is AS in the ball centered $B_{\sqrt{2}}(0)$.



The choice of the Lyapunov function is not unique.

Example 2: damped pendolum

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_2 - \sin(x_1)\end{aligned}$$

Study the stability of $\bar{x} = [0 \quad 0]^T$

Lyapunov candidate function: $V(x) = \underbrace{(1-\cos(x_1))}_{\text{potential en.}} + \underbrace{x_2^2/2}_{\text{kinetic en.}}$.

$$\dot{V} = \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x) = \sin(x_1)x_2 + x_2(-x_2 - \sin(x_1)) = -x_2^2$$

$\dot{V}$ is negative semi definite in $\mathbb{R}^2$ (and then in $B_{2\pi(0)}$) $\implies \bar{x} = 0$ is stable.

Physical intuition tells us the equilibrium is AS but the chosen Lyapunov function certifies only stability

Example 3 Consider the Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= xy - bz\end{aligned}$$

where σ, r and b are positive constants. Consider the folowing Lyapunov function

$$V = \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2$$

where α_1, α_2 and α_3 are positive constants to be determined. It is clear that V is positive definite on $\mathbb{R}^3$ and is radially unbounded.

$$\dot{V} = -2\alpha_1\sigma x^2 - 2\alpha_2 y^2 - 2\alpha_3 b z^2 + xy(2\alpha_1\sigma + 2r\alpha_2) + (2\alpha_3 - 2\alpha_2)xyz$$

If we choose $\alpha_1 = \alpha_2 = 1$, $\alpha_1 = \frac{1}{\sigma}$ and $r < 1$ then $\dot{V}$ becomes

$$\begin{aligned}\dot{V} &= -2\left(x^2 + y^2 + 2bz^2 - (1+r)xy\right)\\ &= -2\left[\left(x - \tfrac{1}{2}(1+r)y\right)^2 + \left(1 - \left(\tfrac{1+r}{2}\right)^2\right)y^2 + bz^2\right]\end{aligned}$$

Since $0 < r < 1$ it folows that $0 < \frac{1+r}{2}$ and therefore $\dot{V}$ is negetive definite on the entire space $\mathbb{R}^3$.

This implies that the origin is **globally asymptotically stable.**

3.3.5 Lyapunov instability theorem

Example

Example: Study the stability of $\bar{x} = 0$

$$\begin{cases}\dot{x}_1 = 2x_1 + x_1(x_1^2 + x_2^4)\\ \dot{x}_2 = -2x_1 + x_2(x_1 + x_2^4)\end{cases} \tag{3.4}$$

We aim to study the stability of the equilibrium point $\bar{x} = 0$.

Step 1: Linearization around $\bar{x} = 0$

To linearize the system, we retain only the linear terms in x_1 and x_2, ignoring higher-order nonlinear terms.

$$\begin{cases} \dot{x}_1 \approx 2x_1 \\ \dot{x}_2 \approx -2x_1 \end{cases} \tag{3.5}$$

However, according to the provided example, the linearized system is:

$$\begin{cases} \dot{x}_1 = 2x_2 \\ \dot{x}_2 = -2x_1 \end{cases} \tag{3.6}$$

Eigenvalues: $\lambda = \pm 2j$ The linearized system has purely imaginary eigenvalues, indicating a center. Therefore, we cannot conclude the stability of the nonlinear system using the linearized system (Lyapunov's indirect method).

Step 2: Lyapunov Direct Method

We consider the candidate Lyapunov function:

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \tag{3.7}$$

This function is positive definite and radially unbounded in $\mathbb{R}^2$.

Now compute the time derivative $\dot{V}$ along the trajectories of the original nonlinear system:

$$\begin{aligned} \dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1\left(2x_1 + x_1(x_1^2 + x_2^4)\right) + x_2\left(-2x_1 + x_2(x_1 + x_2^4)\right) \\ &= 2x_1^2 + x_1^2(x_1^2 + x_2^4) - 2x_1x_2 + x_2^2(x_1 + x_2^4) \end{aligned}$$

This expression can be simplified and grouped as:

$$\dot{V}(x) = (x_1^2 + x_2^2)(x_1^2 + x_2^4)$$

Observation:

- $V(x) > 0$ for all $x \neq 0$, and $V(0) = 0$
- $\dot{V}(x) > 0$ for all $x \neq 0$, and $\dot{V}(0) = 0$
- Thus, $V(x)$ is strictly increasing along trajectories

Conclusion: The equilibrium $\bar{x} = 0$ is **unstable**.

3.3.6 Invariance Principle: Krasovskii-Lasalle LaSalle's Theorem

LaSalle's invariance principle is a tool for assessing asymptotic stability properties of $\bar{x} = 0$ for $\dot{x} = f(x)$ when $\dot{V}(x)$ is only semi-definite

Theorem: LaSalle's Invariance Principle

Let $\dot{x} = f(x)$ be a be a system with a compact positively invariant set Ω and let $V_{LaSalle}(x)$ be a continuously differentiable function with

$$\dot{V}_{LaSalle}(x) \leq 0 \tag{3.8}$$

for all $x \in \Omega$. Further, let N denote the set of all points $x \in \Omega$ with

$$\dot{V}_{LaSalle}(x) = 0 \tag{3.9}$$

and let M denote the largest invariant set in N. In this case all solutions $x(t)$ that start within Ω tend to the set M for $t \to \infty$.

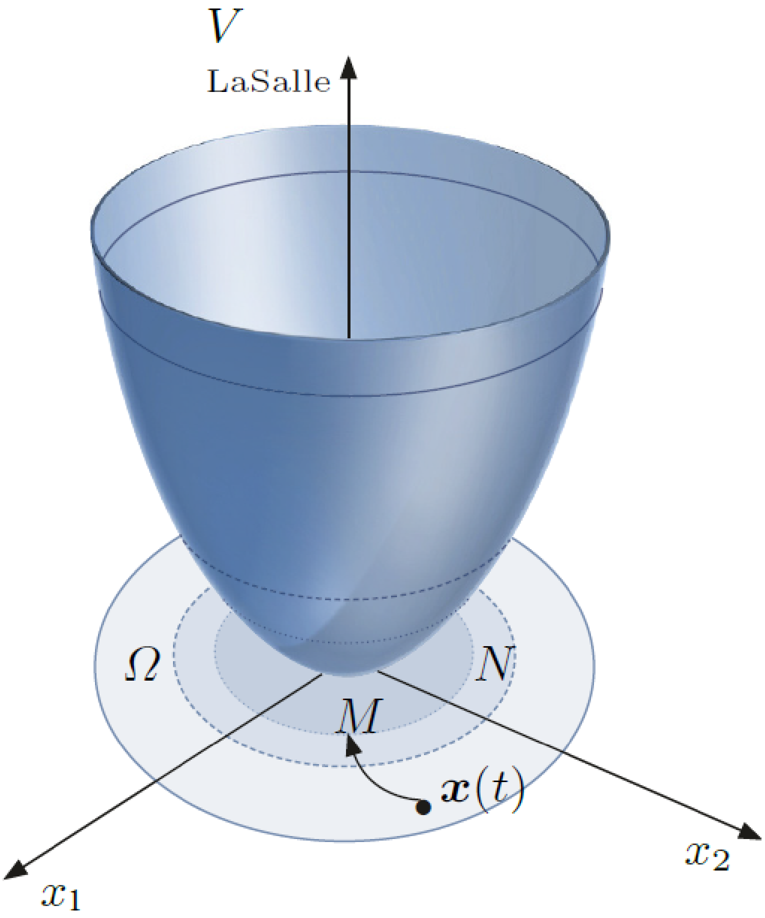


Figure 3.6: LaSalle's Invariance Principle

Remarks

- The theorem provides sufficient conditions for Ω to be a region of attraction for the set M
- Notable case: when $M = \{0\}$ the theorem gives a region of attraction (asymptotic stability) for the equilibrium state $\bar{x} = 0$.

Example: Consider the system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \end{cases} \tag{3.10}$$

Consider the Lyapunov function

$$V(x) = \underbrace{\frac{g}{l}(1-\cos x_1)}_{\text{potential en.}} + \underbrace{\frac{1}{2}x_2^2}_{\text{kinetic en.}}$$

- V(0)=0
- V(x)>0 if $x_1 \in (-2\pi, 2\pi)$

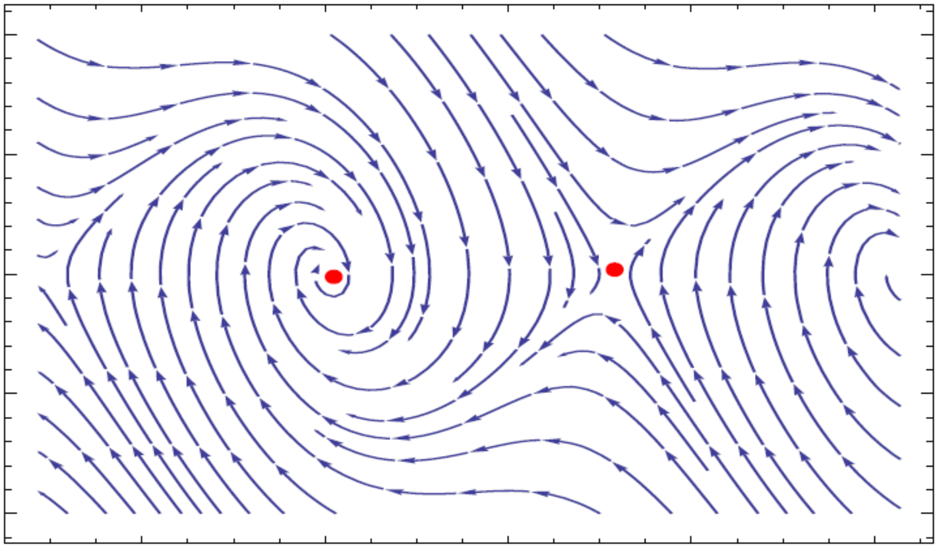
Then V is positive definite in $B_2\pi(0)$

$$\dot{V}(x) = \frac{g}{l}\sin x_1 \dot{x}_1 + x_2\dot{x}_2 = -\frac{k}{m}x_2^2 \leq 0$$

$\dot{V}$ is negative semi definite in $\mathbb{R}^2$ (and then in $B_{2\pi(0)}$)$\Rightarrow x_e = 0$ is stable Then $S := \{(x_1, x_2)|x_2 = 0\}$, i.e., x_1 can be anything and S is the x_1-axis.

Q. what is the largest invariant set ?

1. $x_2 \equiv 0 \Longrightarrow \dot{x}_2 \equiv 0$.
2. $\dot{x}_2 \equiv 0 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}.0 = -\frac{g}{l}\sin(x_1)$



3.3.7 Lyapunov theory for LTI systems

Let $x = \bar{x}$ be an equilibrium for the autonomous nonlinear system

$$\dot{x} = f(x) \tag{3.11}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a continuously differentiable function and $\mathcal{D}$ is a neighborhood of $\bar{x}$. Let

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=\bar{x}}$$

Then:

- $\bar{x}$ is **aymptotically stable** if $\text{Re}(\lambda_i){<}0$ for all eigenvalues of A.
- $\bar{x}$ is **stable** if $\text{Re}(\lambda_i){<}0$ and $\text{Re}(\lambda_i){=}0$ for one of the eigenvalues of A.
- $\bar{x}$ is **unstable** if $\text{Re}(\lambda_i){>}0$ for one or more of the eigenvalues of A.
- In linear systems, *local stability* $\iff$ *global stability.*
- In nonlinear systems, this is not true.

Lyapunov's Indirect Method offers an alternate way to determine the stability of a system by considering the properties of the linearization of the system around the origin.

Review: Positive Definite Matrices

Symmetric matrix $M = M^T$ is

1. **positive definite** (pd) if $x^T M x > 0$, $\forall x \neq 0$.
2. **positive semi-definite** (psd) if $x^T M x \geq 0$, $\forall x \in \mathbb{R}^n$.

Lemma:

- $M = M^T > 0 \iff \lambda_i(M) > 0$
- $M = M^T \geq 0 \iff \lambda_i(M) \geq 0$

Properties of the quadratic function $x^T M x$

From (1) and (2) one has

- if $M > 0$, $V(x) = x^T M x$ is a positive definite (pd) function.
- if $M \geq 0$, $V(x) = x^T M x$ is a positive semi-definite (pd) function.

Lyapunov functions for LTI systems

For linear system $\dot{x} = Ax$.
Consider as Lyapunov candidate function $V(x) = x^T P x$, $P = P^T > 0$.

- $V(x)$ is quadratic, gpd and radially unbounded.
- $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x$

If $A^T P + PA < 0$, i.e. there is $Q > 0$ symmetric such that

$$A^T P + PA = -Q \tag{3.12}$$

then $\dot{V}$ is globally negative definite and by the second Lyapunov method the origin is globally asymptotically stable.

$A^T P + PA = -Q$ is called Lyapunov equation

Remarks

- For LTI systems it is enough to consider quadratic Lyapunov functions.
- Algorithm:
  - choose $Q > 0$ (e.g. Q=I)
  - solve $A^T P + PA = -Q$ (linear systems in the entries of the symmetric matrix P)
  - The LTI system is AS if and only if $P > 0$

Example:

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix}}_{A} x$$

Eigenvalues of $A : \{-1, -3\} \implies$ (global) asymptotic stability.
Choose $Q = Q^T = I_{2\times 2}$. Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, where $p_{12} = p_{21}$.

Solve the Lyapunov equation $A^T P + PA = -Q$

$$\begin{aligned} \begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} -2p_{11} & -4p_{12} + 4p_{11} \\ -4p_{12} + 4p_{11} & 8p_{12} - 6p_{22} \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

Solving for p_{11}, p_{12} and p_{22} gives

$$\begin{cases} 2p_{11} &= -1 \\ -4p_{12} + 4p_{11} &= 0 \\ 8p_{12} - 6p_{22} &= -1 \end{cases}$$

Solving the linear systems one gets

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{bmatrix} > 0$$

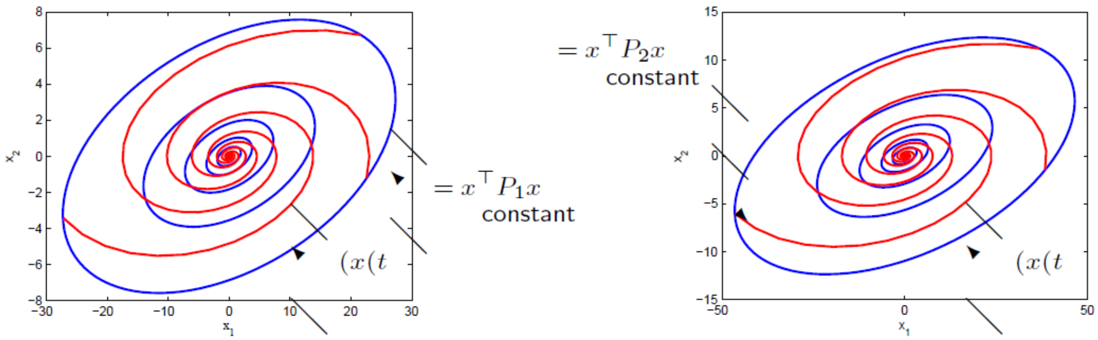
Since $P > 0 \Longrightarrow$ the systems is AS.

Example

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & -16 \\ 1 & -2 \end{bmatrix}}_{A} x \quad \text{Eigenvalues of A} : \{-1 \pm j\sqrt{15}\}$$

Solve $PA + A^T P = -Q$ for P:

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} 0.33 & -0.5 \\ -0.5 & 4.25 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.41 & -0.19 \\ -0.19 & 0.11 \end{bmatrix} \Rightarrow P_2 = \begin{bmatrix} 0.12 & -0.21 \\ -0.21 & 1.67 \end{bmatrix}$$



- any choice of $Q > 0$ gives $P > 0$ (since A is strictly stable)
- but not every $P > 0$ gives $Q > 0$

Chapter 4

Control theory

Given a physical system that we want to control and a desired behavior that we want the system to achieve, control consists in designing control laws such that the system subjected to these laws—i.e., the closed-loop system—exhibits the desired behavior. However, this procedure is only possible if the system in question is controllable; otherwise, the uncontrollable modes must be stable.

4.1 Controllability of Continuous Systems

One of the main goals of control theory is to design control laws such that a system evolves according to a predetermined objective. It is necessary, therefore, for the system to be controllable. Intuitively, the property of controllability means that one can drive the system from an initial state to another using an open-loop control. Conversely, non-controllability means that certain states are unreachable, regardless of the control applied.

4.1.1 Controllability of Linear Systems

In the case of a linear control system:

$$\dot{x} = Ax + Bu \tag{4.1}$$

$$y = Cx \tag{4.2}$$

where $A_{n\times n}$ is the state matrix, $x \in \mathbb{R}^n$ is the state vector, $B_{n\times m}$ is the control matrix, u belongs to the set of admissible controls U, $C_{p\times n}$ is the output matrix, and $y \in \mathbb{R}^p$ are the system outputs.

Definition 4.1 (Definition 4.1)**.** A system given by (4.1) is said to be **controllable** if, for any pair $(x_0, x_d) \in \mathbb{R}^n$, there exists a finite time T and a control input u defined on $[0, T]$, such that the system evolves from the initial state $x(0) = x_0$ to the desired state $x(T) = x_d$.

Kalman Controllability Criterion

There exists an algebraic characterization of the controllability of linear systems due to Kalman.

Theorem 4.1

The linear system (4.1) is controllable if and only if the controllability matrix

$$C = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \tag{4.3}$$

has full rank n. In this case, the pair (A, B) is said to be controllable.

In the case of a controllable system, one seeks to design controllers that render the origin asymptotically stable. One way to achieve this is through state feedback control.

Stabilization via State Feedback

A linear state feedback or linear regulator for system (4.1) is a control law of the form:

$$u(t) = -Kx(t) \tag{4.4}$$

where $K \in \mathbb{R}^{m \times n}$ is the feedback gain matrix.

The gain matrix K can be determined in several ways, for example by pole placement.

When the value of $u(t)$ at time t depends only on $x(t)$, the feedback is static, as illustrated in Figure 4.1.

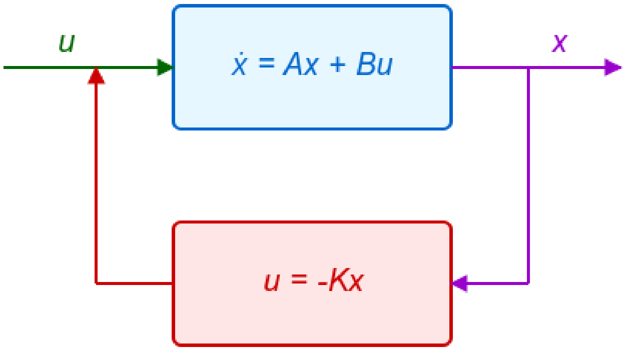


Figure 4.1: Space state feedback stabilization.

Pole Placement Design

When the system is controllable, the pole placement method consists in determining a feedback law of the form $u = -Kx$ such that:

$$\sigma(A - BK) = \sigma_d$$

where $\sigma(A - BK)$ is the spectrum (set of eigenvalues) of $(A - BK)$, and σ_d is the desired spectrum.

The difficulty of this approach lies in selecting a desired spectrum since the eigenvalue placement affects system performance. For instance, if the poles are located too far to the left in the complex plane, the system becomes fast but sensitive to noise and modeling errors. Conversely, if they are placed too close to the imaginary axis, the system becomes too slow and may not meet the desired performance.

Note: On the negative half-plane, the values of K can become very large and may cause saturation problems that could lead to instability.

State Reconstruction:

The state estimator (or observer) is a system that reconstructs the states from known quantities, i.e., inputs and outputs of the process, such that:

$$\lim_{t\to\infty}(\hat{x}(t)-x(t))=0$$

The same operation is only possible if the system is observable, meaning the Kalman observability matrix:

$$\mathbf{O}=\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

is of rank n. We then say that the pair (A, C) is observable.
For linear systems, Luenberger proposes the following observer structure:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y-\hat{y}) \\ \hat{y} = C\hat{x} \end{cases} \tag{4.5}$$

which corresponds to a simulation of the system constantly corrected by the difference between the observed output and the reconstructed output.
If we define $\tilde{x}=\hat{x}-x$, it follows that:

$$\dot{\tilde{x}} = (A-LC)\tilde{x}$$

The problem is to determine L such that $\lim_{t\to\infty}(\hat{x}(t)-x(t))=0$ for initial conditions $x(0)\neq\hat{x}(0)$.

The analysis is immediate; we will have $\tilde{x}\to 0$ if L is chosen such that $(A-LC)$ is stable. One encounters a dual problem to that of stabilization. Thus, all techniques used for controllability can be applied for state reconstruction by replacing A with A^T, B with B^T, and L with L^T, with the exception that the choice of the observer's spectrum must be several times faster.

Much faster than that of the control law to ensure convergence of the estimation before applying the command. However, if this spectrum is chosen too far in the negative half-plane, the observer will be too fast with a wider bandwidth, making it more sensitive to measurement noise.

Exercise:

Consider the system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

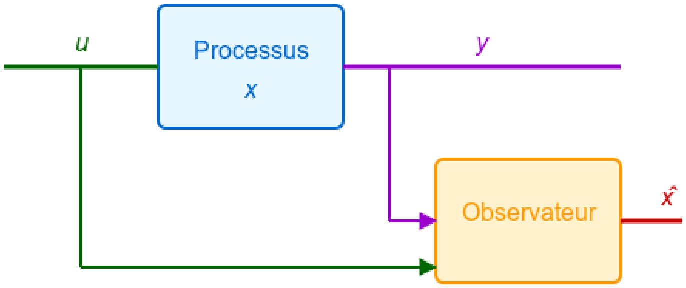


Figure 4.2: Reconstructor of State

Determine a control law of the form:

$$u(x) = -Kx + v$$

such that the closed-loop poles are located at -1 and -2, and compute the input v such that $x_1(t) \to 3$ as $t \to \infty$.

We rewrite the system in state-space form:

$$\dot{x} = Ax + Bu \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Controllability: To verify controllability, compute:

$$AB = AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad C = [B \; AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det(C) = -1 \neq 0 \quad \Rightarrow \quad \text{The system is controllable.}$$

Stabilization via State Feedback: We use a state feedback of the form:

$$u = -Kx = -[k_1 \quad k_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then the closed-loop system becomes:

$$\dot{x} = (A - BK)x$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

The characteristic polynomial of the closed-loop matrix is:

$$|\lambda I - (A - BK)| = \left| \begin{bmatrix} \lambda & -1 \\ k_1 & \lambda + k_2 \end{bmatrix} \right| = \lambda(\lambda + k_2) + k_1 = \lambda^2 + k_2\lambda + k_1$$

Matching this with the desired polynomial:

$$(\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2 \quad \Rightarrow \quad \begin{cases} k_2 = 3 \\ k_1 = 2 \end{cases}$$

Thus, the stabilizing control law is:

$$u(x) = -2x_1 - 3x_2$$

Tracking a Constant Reference: To ensure that $x_1(t) \to 3$ as $t \to \infty$, we add a constant input v:

$$u(x) = -2x_1 - 3x_2 + v$$

Substituting into the system yields:

$$\dot{x} = (A - BK)x + Bv \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

At steady state:

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0, \quad \dot{x}_2 = 0 \Rightarrow -2x_1 + v = 0$$

If we want $x_1 = 3$ at steady state, then:

$$-2 \cdot 3 + v = 0 \Rightarrow v = 6$$

Final Control Law: The full control law ensuring both stabilization and reference tracking is:

$$u(x) = -2x_1 - 3x_2 + 6$$

Remark: This control law assumes full state feedback, i.e., all components of the state vector x are available for measurement. In practice, this may not always be the case due to technological or economic constraints. In such situations, one typically designs an observer to reconstruct the unmeasurable states.

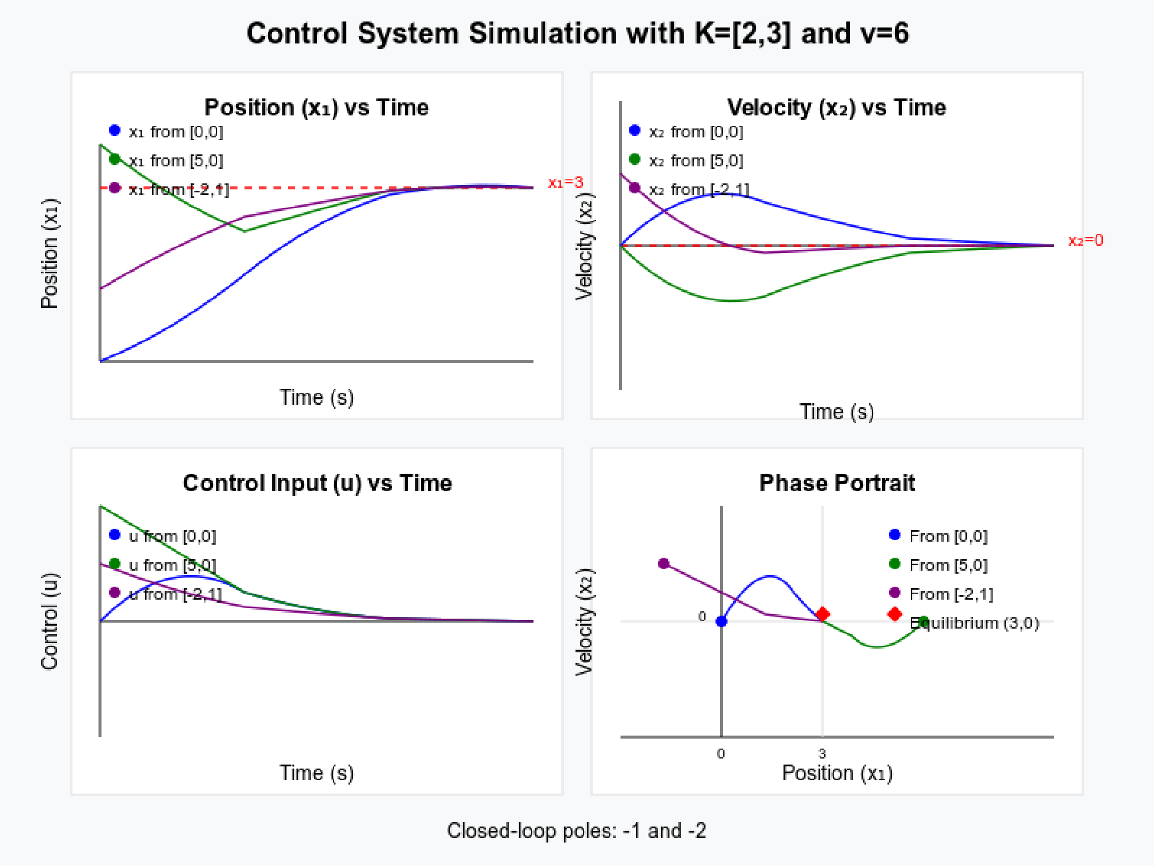


Figure 4.3: Simation of exerise

4.2 Controllability concepts of nonlinear systems.

4.3 Exact Feedback Linearization

When a system is transformed through a change of coordinates, some of its properties remain unchanged. For example, if a system is unstable, then the transformed system is also unstable; if a system is controllable, then the transformed system is also controllable. However, some systems may appear nonlinear in certain coordinates, while they can appear in a linear form in other coordinates under a suitable feedback.

Moreover, it is useful, when possible, to approximate certain dynamics with other, easier-to-study dynamics. This procedure is carried out through diffeomorphisms.

Diffeomorphism

A diffeomorphism is a nonlinear coordinate change of the form $z = \Phi(x)$, where Φ is a vector function:

$$\Phi(x) = \begin{bmatrix} \Phi_1(x_1, \ldots, x_n) \\ \Phi_2(x_1, \ldots, x_n) \\ \vdots \\ \Phi_n(x_1, \ldots, x_n) \end{bmatrix}$$

with the following properties: - $\Phi(x)$ is a bijective application - $\Phi(x)$ and Φ^{-1} are differentiable applications.
If these properties are verified for all $x \in \mathbb{R}^n$, then Φ is a global diffeomorphism; otherwise, Φ is a local diffeomorphism.

Proposition

If the Jacobian matrix of Φ, evaluated at point $x = x_0$, is non-singular, then $\Phi(x)$ is a local diffeomorphism.

4.3.1 Exact Input-State Feedback Linearization

Given a control-affine system:

$$\dot{x} = f(x) + g(x)u$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}$, f and g are assumed to be regular.

The goal is to find a transformation:

$$z = \Phi(x)$$

and a control:

$$u(x) = \alpha(x) + \beta(x)v$$

such that in the new coordinates, the initial system becomes:

$$\dot{z} = Az + Bv$$

where A and B are constant matrices.

Thus, the idea is to transform system (4.14) through a coordinate change $z = \Phi(x)$ such that in the new coordinates z, system (4.14) is expressed by:

$$\dot{z} = Az + \Psi(z) + Bu$$

with:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \Psi(z) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \psi(z) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

More precisely:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ z_4 \\ \vdots \\ z_n \\ \psi(z_1, z_2, \dots, z_n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

If such a transformation $z = \Phi(x)$ exists, then a control change can be applied:

$$u(x) = -\Psi(z_1, \dots, z_n) + v$$

such that the transformed system appears linear:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= v \end{aligned}$$

v is a new linear control that can be determined by a linear synthesis method such as:

$$v = -Kz$$

In closed loop, the z-system becomes:

$$\dot{z} = (A - BK)z$$

K is a gain matrix chosen such that $(A - BK)$ is stable.

In this case, we say that the transformation $z = \Phi(x)$ and the control change $u(x)$ have allowed an exact linearization of system (4.14) by feedback.

Question : How to obtain the coordinate change $z = \Phi(x)$?

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \\ \Phi_3(x) \\ \vdots \\ \Phi_n(x) \end{bmatrix}$$

Answer

We can demonstrate that a possible (and not unique) transformation is given by:

$$\Phi(x) = \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \\ \Phi_3(x) \\ \vdots \\ \Phi_n(x) \end{bmatrix} = \begin{bmatrix} \Phi_1(x) \\ L_f\Phi_1(x) \\ L_f\Phi_2(x) \\ \vdots \\ L_f\Phi_{n-1}(x) \end{bmatrix} = \begin{bmatrix} \Phi_1(x) \\ L_f\Phi_1(x) \\ L_f^2\Phi_1(x) \\ \vdots \\ L_f^{n-1}\Phi_1(x) \end{bmatrix}$$

Thus, finding $\Phi(x)$ comes down to finding $\Phi_1(x)$, and by recursion we find the other components $\Phi_2(x), \ldots, \Phi_n(x)$.
It remains now to determine $\Phi_1(x)$.
One way to determine $\Phi_1(x)$ is to solve the system:

$$\frac{\partial \Phi_1(x)}{\partial x} = (0\;0\;\ldots\;1)C_{fg}^{-1}(x)$$

where $C_{fg}^{-1}(x)$ is the inverse of the controllability matrix (assumed invertible if the system is controllable).
Consequently, $\frac{\partial \Phi_1(x)}{\partial x}$ can be equal to the last row of the inverse controllability matrix.

Proof

$$\dot{x} = f(x) + g(x)u \quad x \in \mathbb{R}^n; u \in \mathbb{R}$$

We seek

$$z = \Phi(x) \rightarrow \dot{z} = Az + \Psi(z) + Bu \tag{4.19}$$

With:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 & 0 \end{bmatrix}, \Psi(z) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Psi(z) \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_n(x) \end{bmatrix}$$

$$\begin{aligned} z &= \Phi(x) \\ \dot{z} &= \frac{\partial \Phi}{\partial x}\dot{x} \\ &= \frac{\partial \Phi}{\partial x}f(x) + \frac{\partial \Phi}{\partial x}g(x)u \end{aligned}$$

From (4.19), we can write:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \vdots \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= \Psi(z) + u \end{aligned} \tag{4.20}$$

Consequently, we get :

$$\frac{\partial \phi}{\partial x} f(x) = Az + \Psi(z) \tag{4.21}$$

and

$$\frac{\partial \phi}{\partial x} g(x) = B \tag{4.22}$$

From (4.22), we can write:

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{4.23}$$

For the first row, we have:

$$\frac{\partial \phi_1}{\partial x_1} g_1(x) + \frac{\partial \phi_1}{\partial x_2} g_2(x) + ... + \frac{\partial \phi_1}{\partial x_n} g_n(x) = L_g \phi_1(x) = 0 \tag{4.24}$$

And by iterations, we can deduce the following relations:

$$L_g \phi_i(x) = 0$$

for $i = 1, 2, ..., n-1$

$$L_g \phi_n(x) = 1$$

From (4.21):

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ \Psi(z) \end{bmatrix} = \begin{bmatrix} \phi_2(z) \\ \phi_3(z) \\ \vdots \\ \Psi(\phi(x)) \end{bmatrix}$$

Similarly:

$$L_f \phi_i(x) = \phi_{i+1}(x) \quad i = 1, ..., n-1 \tag{4.26}$$

$$L_f \phi_n = \Psi(\phi(x))$$

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ L_f \phi_1(x) \\ L_f \phi_2 \\ \vdots \\ L_f \phi_{n-1}(x) \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ L_f \phi_1(x) \\ L_f L_f \phi_1(x) \\ \vdots \\ L_f^{n-1} \phi_1(x) \end{bmatrix} = \begin{bmatrix} \phi_1 \\ L_f \phi_1(x) \\ L_f^2 \phi_1(x) \\ \vdots \\ L_f^{n-1} \phi_1(x) \end{bmatrix}$$

From (4.25) and (4.26), we can write:

$$[L_g \phi_1(x) \; L_g \phi_2(x) \; \ldots \; L_g \phi_{n-1}(x) \; L_g \phi_n] = [0 \; 0 \; \ldots \; 0 \; 1]$$

$$[\frac{\partial \phi_1}{\partial x} g(x)\ L_g L_f \phi_1(x)\ \dots\ L_g L_f^{n-2}\phi_1\ L_g L_f^{n-1}\phi_1] = [0\ 0\ \dots\ 0\ 1] \tag{4.27}$$

$$[\frac{\partial \phi_1}{\partial x} ad_f^0 g(x)\ \frac{\partial \phi_1}{\partial x} ad_f^1 g(x)\ \dots\ \frac{\partial \phi_1}{\partial x} ad_f^{n-2} g(x)\ \frac{\partial \phi_1}{\partial x} ad_f^{n-1} g(x)] = [0\ 0\ \dots\ 0\ 1]$$

To write the last line of (4.27), we need the following reminders:

Reminder 1: Leibniz-Jacobi Identity:

$$L_{[f,g]}h = L_g L_f h - L_f L_g h \Rightarrow L_g L_f h = L_f L_g h + L_{[f,g]}h$$

Reminder 2: Successive Lie brackets:

$$\begin{array}{rcl} ad_f^0 g(x) & = & g(x) \\ ad_f^1 g(x) & = & [f,g] \\ ad_f^2 g(x) & = & [f,[f,g]] \\ ad_f^k g(x) & = & [f, ad_f^{k-1} g(x)] \end{array}$$

Taking into account these reminders and (4.25):

$$L_g L_f \phi_1 = L_f L_g \phi_1 + L_{[f,g]}\phi_1 = 0 + L_{ad_f^1 g(x)}\phi_1 = \frac{\partial \phi_1}{\partial x} ad_f^1 g(x) \tag{4.28}$$

From (4.27):

$$[\frac{\partial \phi_1}{\partial x}][ad_f^0 g(x)\ ad_f^1 g(x)\ \dots\ ad_f^{n-1} g(x)] = [0\ 0\ \dots\ 1]$$

$$\frac{\partial \phi_1}{\partial x} C_{fg}(x) = [0\ \dots\ 1]$$

$$\frac{\partial \phi_1}{\partial x} = [0\ \dots\ 1] C_{fg}^{-1} = q(x)$$

which is the last row of the inverse controllability matrix.

Therefore, we need to solve $\frac{\partial \phi_1}{\partial x} = q(x)$ to find $\phi_1(x)$.

Remark: Solving the system of partial differential equations is not easy, and moreover, the solution is not unique.

Definition

A set of m independent vectors $\{g_1, \dots, g_m\}$ in Ω is said to be involutive if for all i and j, the bracket $[g_i, g_j]$ is a linear combination of the vectors $g_1, \dots, g_m$, i.e., there exist functions α_{ij}^k defined in Ω such that:

$$[g_i, g_j] = \sum_{k=1}^{m} \alpha_{ij}^k g_k$$

In other words, if for all f and g in Ω, then $[f,g]$ belongs to Ω (Ω is closed under the Lie bracket).

Definition

System (4.14) is input-state linearizable in a domain D if and only if:

1. The rank of the controllability matrix $C_{fg} = \{g, ad_f g, \ldots, ad_f^{n-1} g\}$ is equal to n for all $x \in D$.
2. The system $\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is involutive in D.

Exercise: Determine the coordinate change and the linearizing feedback control for the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ 10\sin x_1 + x_3 \\ -10x_2 - 10x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

Solution:

$$f(x) = \begin{bmatrix} x_2 \\ 10\sin x_1 + x_3 \\ -10x_2 - 10x_3 \end{bmatrix}$$

, $g(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Controllability:

$$rg\ C = rg[g\ ad_f g\ ad_f^2 g] = n$$

$$ad_f g = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = -\frac{\partial f}{\partial x} g = -\begin{bmatrix} 0 & 1 & 0 \\ 10\cos x_1 & 0 & 1 \\ 0 & -10 & -10 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 100 \end{bmatrix}$$

$$ad_f^2 g = [f, ad_f g] = \frac{\partial ad_f g}{\partial x} f - \frac{\partial f}{\partial x} ad_f g = -\begin{bmatrix} 0 & 1 & 0 \\ 10\cos x_1 & 0 & 1 \\ 0 & -10 & -10 \end{bmatrix} \begin{bmatrix} 0 \\ -10 \\ 100 \end{bmatrix} = \begin{bmatrix} 10 \\ -100 \\ 900 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 10 \\ 0 & -10 & -100 \\ 10 & 100 & 900 \end{bmatrix}$$

$$\det = 10 \cdot 100 = 1000 \Rightarrow$$

controllable

Involutivity: Is the set $\{g(x), ad_f g\}$ involutive?

$$[g, ad_f g] = \frac{\partial ad_f g}{\partial x} g - \frac{\partial g}{\partial x} ad_f g = 0 = 0g + 0ad_f g \Rightarrow \text{the set is involutive}$$

$$\frac{\partial \phi_1}{\partial x} = [0\ 0\ 1]C^{-1} = [0\ 0\ 1] \begin{bmatrix} 1 & 1 & 0.1 \\ -1 & -0.1 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \phi_1}{\partial x} = [0.1\ 0\ 0]$$

$$\frac{\partial \phi_1}{\partial x_1} = 0.1 \Rightarrow \phi_1 = 0.1x_1 + c_1$$

Particularly if we choose

$$c_1 = 0 \Rightarrow \phi_1 = 0.1x_1$$

Consider the following coordinate change:

$$z_1 = 0.1x_1 = \phi_1$$

$$z_2 = L_f\phi_1 = \frac{\partial \phi_1}{\partial x} f = [0.1\ 0\ 0] \begin{bmatrix} x_2 \\ 10\sin x_1 + x_3 \\ -10x_2 - 10x_3 \end{bmatrix} = 0.1x_2 = \phi_2$$

$$z_3 = L_f\phi_2 = \frac{\partial \phi_2}{\partial x} f = [0\ 0.1\ 0] \begin{bmatrix} x_2 \\ 10\sin x_1 + x_3 \\ -10x_2 - 10x_3 \end{bmatrix} = \sin x_1 + 0.1x_3$$

$$z_1 = 0.1x_1 \Rightarrow x_1 = 10z_1$$

$$z_2 = 0.1x_2 \Rightarrow x_2 = 10z_2$$

$$z_3 = \sin x_1 + 0.1x_3 \Rightarrow x_3 = 10[z_3 - \sin 10z_1]$$

By differentiating these variables, we obtain a partially linear system:

$$\begin{aligned} \dot{z}_1 &= 0.1\dot{x}_1 = 0.1x_2 = z_2 \\ \dot{z}_2 &= 0.1\dot{x}_2 = 0.1(10\sin x_1 + x_3) = \sin x_1 + 0.1x_3 = z_3 \\ \dot{z}_3 &= \dot{x}_1 \cos x_1 + 0.1\dot{x}_3 = x_2\cos x_1 + 0.1[(-10x_2 - 10x_3) + 10u] \\ &= x_2 \cos x_1 - x_2 - x_3 + u \\ &= 10z_2\cos(10z_1) - 10z_2 - 10(z_3 - \sin z_1) + u \end{aligned}$$

The control law $u(z)$ is chosen such that:

$$\begin{aligned} u(z) &= -\Psi(z) + l_1z_1 + l_2z_2 + l_3z_3 && (4.1) \\ &= -10z_2\cos(10z_1) + 10z_2 + 10(z_3 - \sin z_1) + l_1z_1 + l_2z_2 + l_3z_3 && (4.2) \end{aligned}$$

Once $u(z)$ is determined, we must return to $u(x)$ through the inverse coordinate transformation because $u(x)$ is the actual control of the original nonlinear system.

$$u(x) = -x_2\cos x_1 + x_2 + x_3 + 0.1l_1x_1 + 0.1l_2x_2 + l_3(\sin x_1 + 0.1x_3) \tag{4.3}$$

4.4 Coordinate Transformation and Linearization

Consider the following change of coordinates:

$$\begin{aligned}
z_1 &= 0.1x_1 = \phi_1 \\
z_2 &= L_f\phi_1 = \frac{\partial \phi_1}{\partial x} f = \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ 10\sin x_1 + x_3 \\ -10x_2 - 10x_3 \end{bmatrix} = 0.1x_2 = \phi_2 \\
z_3 &= L_f\phi_2 = \frac{\partial \phi_2}{\partial x} f = \begin{bmatrix} 0 & 0.1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ 10\sin x_1 + x_3 \\ -10x_2 - 10x_3 \end{bmatrix} = \sin x_1 + 0.1x_3
\end{aligned} \tag{4.4}$$

From these relations, we have:

$$\begin{aligned}
z_1 &= 0.1x_1 \Rightarrow x_1 = 10z_1 \\
z_2 &= 0.1x_2 \Rightarrow x_2 = 10z_2 \\
z_3 &= \sin x_1 + 0.1x_3 \Rightarrow x_3 = 10[z_3 - \sin(10z_1)]
\end{aligned} \tag{4.5}$$

By differentiating these variables, we obtain a partially linear system:

$$\begin{aligned}
\dot{z}_1 &= 0.1\dot{x}_1 = 0.1x_2 = z_2 \\
\dot{z}_2 &= 0.1\dot{x}_2 = 0.1(10\sin x_1 + x_3) = \sin x_1 + 0.1x_3 = z_3 \\
\dot{z}_3 &= \dot{x}_1 \cos x_1 + 0.1\dot{x}_3 = x_2 \cos x_1 + 0.1[(-10x_2 - 10x_3) + 10u] \\
&= x_2 \cos x_1 - x_2 - x_3 + u \\
&= 10z_2 \cos(10z_1) - 10z_2 - 10(z_3 - \sin z_1) + u
\end{aligned} \tag{4.6}$$

Let's define $\Psi(z) = 10z_2 \cos(10z_1) - 10z_2 - 10(z_3 - \sin z_1)$

The control law $u(z)$ is chosen such that:

$$\begin{aligned}
u(z) &= -\Psi(z) + l_1 z_1 + l_2 z_2 + l_3 z_3 \\
&= -10z_2 \cos(10z_1) + 10z_2 + 10(z_3 - \sin z_1) + l_1 z_1 + l_2 z_2 + l_3 z_3
\end{aligned} \tag{4.7}$$

Once $u(z)$ is determined, we must return to $u(x)$ through the inverse coordinate transformation because $u(x)$ is the actual control of the original nonlinear system.

$$u(x) = -x_2 \cos x_1 + x_2 + x_3 + 0.1l_1 x_1 + 0.1l_2 x_2 + l_3(\sin x_1 + 0.1x_3) \tag{4.8}$$

4.5 Exact Linearization through Input-Output Feedback

4.5.1 Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ 10\sin x_1 + x_3 \\ -10x_2 - 10x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \tag{4.9}$$

with the output

$$y = x_1 \tag{4.10}$$

Let's differentiate y until the input u appears:

$$\begin{aligned} y &= x_1 \\ \dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = 10\sin x_1 + x_3 + u \end{aligned} \tag{4.11}$$

The relative degree associated with this example is 2.

In this case, we can only linearize the subsystem formed by the first two equations by choosing u to eliminate $10\sin x_1$ such that:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 10\sin x_1 + x_3 + u \end{aligned} \tag{4.12}$$

The control u is chosen such that:

$$10\sin x_1 + x_3 + u = l_1 x_1 + l_2 x_2 \tag{4.13}$$

Therefore:

$$u(x) = l_1 x_1 + l_2 x_2 - 10\sin x_1 - x_3 \tag{4.14}$$

In closed loop:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= l_1 x_1 + l_2 x_2 \\ \dot{x}_3 &= -10x_2 - 10x_3 \end{aligned} \tag{4.15}$$

The third dynamic is called the zero dynamics, for which stability must be verified. For this system, it is linear and stable because variable x_2 is bounded and stabilized, and could be viewed as an input.

4.5.2 General Framework

In general, the relative degree associated with a system is the number of successive derivatives of the output required to make the input appear. For a single-input system:

$$\dot{x} = f(x) + g(x)u \tag{4.16}$$

$$y = h(x) \tag{4.17}$$

Let's differentiate the output:

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}(f + gu) \tag{4.18}$$

$$= \frac{\partial h}{\partial x}(f) + \frac{\partial h}{\partial x}(gu) \tag{4.19}$$

$$= L_f h + L_g h u \tag{4.20}$$

If $L_g h \neq 0$, then the relative degree $r = 1$

If $L_g h = 0$, then we continue differentiating:

$$\begin{aligned} \ddot{y} &= \frac{d}{dt} L_f h = \frac{\partial L_f h}{\partial x}\dot{x} \\ &= \frac{\partial L_f h}{\partial x}(f + gu) \\ &= L_f^2 h + L_g L_f h u \end{aligned} \tag{4.21}$$

If $L_g L_f h \neq 0$, then the relative degree $r = 2$

If $L_gL_fh = 0$, then we continue differentiating, at order r:

$$y^{(r)} = L_f^r + L_gL_f^{r-1}hu \tag{4.22}$$

If $L_gL_f^{r-1}h \neq 0$, then the relative degree is r
In this case:

$$\begin{aligned} y &= h(x) \\ \dot{y} &= L_fh(x) \\ \ddot{y} &= L_f^2h(x) \\ &\vdots \\ y^{(r-1)} &= L_f^{r-1}h(x) \end{aligned} \tag{4.23}$$

Definition 4.2. The relative degree associated with system (4.31) in a region $\Omega \subset \mathbb{R}^n$ is given by the integer r such that

$$L_gh(x) = L_gL_fh(x) = \ldots = L_gL_f^{r-2}h(x) = 0 \tag{4.24}$$

$$L_gL_f^{r-1}h(x) \neq 0 \tag{4.25}$$

for all $x \in \Omega$.

The idea of exact linearization through input-output feedback is to generate linear equations between the output y and a certain input v through a diffeomorphism $z = \phi(x)$ consisting of the output and its time derivatives up to order $n-1$ when the relative degree r associated with this system equals n.

$$\begin{aligned} \phi_1(x) &= h(x) \\ \phi_2(x) &= L_fh(x) \\ &\vdots \\ \phi_n(x) &= L_f^{n-1}h(x) \end{aligned} \tag{4.26}$$

The transformed system is written in the form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= L_f^n(\phi^{-1}(z)) + L_gL_f^{n-1}(\phi^{-1}(z))u \end{aligned} \tag{4.27}$$

By choosing u of the form:

$$u = \frac{1}{L_gL_f^{n-1}(\phi^{-1}(z))}(v - L_f^n(\phi^{-1}(z))) \tag{4.28}$$

for $L_gL_f^{n-1}(\phi^{-1}(z)) \neq 0$, the system becomes:

$$\dot{z}_1 = z_2, \dot{z}_2 = z_3, \ldots, \dot{z}_{n-1} = z_n, \dot{z}_n = v \tag{4.29}$$

Note that in this case, the form (4.33) is the same as in (4.17). In fact, when $r = n$, the two linearizations are equivalent. Thus, the application conditions of the second linearization will be the same as for the first.

Obviously, for a relative degree $r < n$, the system is no longer completely linearizable by feedback. In this case, we speak of partial linearization by feedback.

4.6 Partial Linearization by Feedback

When $r < n$, it is only possible to partially linearize a system of the form (4.31) through a diffeomorphism consisting partly of the output $h(x)$ and its successive derivatives up to order $r-1$: $z = \phi_i(x)$ for $1 < i < r$, and completed, by Frobenius' theorem, by $n-r$ other functions: $\eta = \phi_i(x)$ for $r+1 \leq i \leq n$, chosen such that $L_g\phi_i = 0$ for $r+1 \leq i \leq n$.

In the coordinates (z, η), the system equations are:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= L_f^r(z,\eta) + L_g L_f^{r-1}(z,\eta)u \\ \dot{\eta} &= q(z,\eta) \end{aligned} \tag{4.30}$$

with

$$y = z_1 \tag{4.31}$$

This form is called the normal form.

For $L_g L_f^{r-1}(z,\eta) \neq 0$, u can still be chosen of the form:

$$u = \frac{1}{L_g L_f^{r-1}(z,\eta)}(v - L_f^r(z,\eta)) \tag{4.32}$$

In this case, the system takes the form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= v \\ \dot{\eta} &= q(z,\eta) \end{aligned} \tag{4.33}$$

Clearly, it appears that this system is composed of a linear subsystem of dimension r controllable by v and "responsible" for the input-output behavior, and a nonlinear subsystem of dimension $n-r$ whose behavior is not affected by the control.

It follows that the global behavior of the system depends on this internal dynamics, and verifying its stability is an essential step.

In fact, it has been shown that the study of the stability of internal dynamics can be reduced to that of zero dynamics. The latter is obtained when a control u is applied that brings the output y to zero and keeps it there.

In other words, the zero dynamics is given by the system $\dot{\eta} = q(0, \eta)$.

Remark 4.1.

1. When $\dot{\eta} = q(0, \eta)$ is (locally) asymptotically stable, then the associated system is said to be (locally) minimum phase at equilibrium x^*.
2. When $\dot{\eta} = q(0, \eta)$ is unstable, then the associated system is said to be non-minimum phase.

4.7 Lyapunov-based Feedback Design

4.7.1 Model Reference Adaptive Control (MRAC)

The Model Reference Adaptive Control (MRAC) method aims to design a control law for a system in which the system's parameters are unknown, but a reference model's behavior is specified. The goal is to ensure that the output of the controlled system tracks the output of the reference model, even though the system's parameters may change over time.

Let's consider a first-order system described by the following differential equation:

$$\dot{y} = a^* y + u$$

where y is the output, u is the control input, and a^* is an unknown system parameter that we do not know a priori.

4.7.2 Reference Model

We now define a reference model that we want our system to track. This reference model has the following form:

$$\dot{y}_m = -a y_m + r(t)$$

where y_m is the output of the reference model, $a > 0$ is a known positive parameter, and $r(t)$ is a reference signal. The model generates the desired trajectory for the system's output.

4.7.3 Control Objective

The objective is to design a controller that guarantees the tracking error $e(t) = y(t) - y_m(t)$ approaches zero as time tends to infinity, i.e., $e(t) \to 0$. This needs to be achieved **without prior knowledge of a^***.

4.7.4 Ideal Control Law (If a^* Were Known)

If we had perfect knowledge of the system parameter a^*, we would choose a control law of the form:

$$u = -\left(a^* + a\right) y + r(t)$$

This would ensure that the closed-loop system behaves as follows:

$$\dot{y} = -ay + r(t)$$

Now, we consider the tracking error $e(t) = y(t) - y_m(t)$, which would satisfy the following error dynamics:

$$\dot{e} = -ae$$

This implies that the error $e(t)$ decays exponentially to zero over time, and the output $y(t)$ would asymptotically track the reference model $y_m(t)$.

4.7.5 Adaptive Control Design (When a^* is Unknown)

In the case where a^* is unknown, we cannot directly implement the ideal control law. Instead, we use an adaptive control law where we approximate the unknown parameter with an adaptive estimate $k(t)$, so the control input becomes:

$$u = -k(t)y + r(t)$$

The goal now is to design $\dot{k}(t)$ such that the tracking error $e(t)$ tends to zero over time.

4.7.6 Error Dynamics with Adaptive Controller

With the adaptive control law, the error dynamics are as follows:

$$\dot{e} = -ae - (k(t) - a^*)y$$

where $\tilde{k}(t) = k(t) - a^*$ is the estimation error. This equation shows that the error is influenced by both the system dynamics and the error in the parameter estimate $k(t)$.

4.7.7 Lyapunov Analysis for Stability

To analyze the stability of the system, we define a Lyapunov function candidate:

$$V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$$

Now, we compute the time derivative of V:

$$\dot{V} = -ae^2 - \tilde{k}ey + \tilde{k}\dot{\tilde{k}}$$

Since $\dot{\tilde{k}} = \dot{k}$, we substitute $\dot{k} = ey$ to obtain:

$$\dot{V} = -ae^2 + \tilde{k}(ey - ey) = -ae^2$$

Thus, $\dot{V}$ is non-positive, meaning that the Lyapunov function V does not increase over time, which guarantees that the error $e(t)$ and the estimation error $\tilde{k}(t)$ remain bounded.

4.7.8 Stability and Convergence

From the Lyapunov analysis, we conclude the following:

- **Stability**: The system will be stable, meaning that the error $e(t)$ and the estimation error $\tilde{k}(t)$ will remain bounded for all time.
- **Convergence**: If the reference signal $r(t)$ is bounded, the output $y(t)$ will track the reference model $y_m(t)$, and $e(t) \to 0$ as time tends to infinity.
- **Parameter Convergence**: Whether $k(t) \to a^*$ depends on the specific properties of the reference signal $r(t)$. Further analysis of $r(t)$ is required to determine this fully.

Whether $\tilde{k}(t) \to 0$ (i.e., $k(t) \to k^*$) depends on further properties of the reference signal $r(\cdot)$ that are beyond the scope of this lecture.

4.8 Model Reference Adaptive Control for a Second-Order System

Consider a second-order system described by the following dynamics:

$$\dot{y} = a^*y_1 + b^*y_2 + u$$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -a^*y_1 - b^*y_2 + u$$

where y_1 and y_2 are the state variables, u is the control input, and a^* and b^* are unknown parameters.

We want to design a controller to make the system output $y(t)$ track a reference model $y_m(t)$ without knowing the exact values of a^* and b^*.

4.8.1 Reference Model

We define the reference model as:

$$\dot{y}_m = ay_m + by_{m2} + r(t)$$

$$\dot{y}_{m1} = y_{m2}$$

$$\dot{y}_{m2} = -ay_{m1} - by_{m2} + r(t)$$

where $r(t)$ is the reference signal, and a, b are the known parameters of the reference model.

4.8.2 Control Objective

The goal is to design an adaptive controller such that the tracking error $e(t) = y(t) - y_m(t)$ tends to zero as time tends to infinity, i.e., we want $e(t) \to 0$.

4.8.3 Ideal Control Law (If a^* and b^* Were Known)

If we knew a^* and b^*, we would choose the following control law:

$$u = -\left(a^* y_1 + b^* y_2 + ay_m + by_{m2}\right) + r(t)$$

The closed-loop system would then behave as:

$$\dot{y} = ay_m + by_{m2} + r(t) - a^* y_1 - b^* y_2$$

The tracking error $e(t) = y(t) - y_m(t)$ would satisfy the following error dynamics:

$$\dot{e} = -ae_1 - be_2$$

where e_1 and e_2 are the errors in the state variables, and the error dynamics will guarantee exponential convergence to zero.

4.8.4 Adaptive Control Design (When a^* and b^* are Unknown)

In the case where a^* and b^* are unknown, we introduce adaptive parameters $k_1(t)$ and $k_2(t)$ to estimate a^* and b^*. The adaptive control law becomes:

$$u = -k_1(t)y_1 - k_2(t)y_2 + ay_m + by_{m2} + r(t)$$

Now, we need to design $\dot{k_1}(t)$ and $\dot{k_2}(t)$ such that the tracking error $e(t)$ tends to zero over time.

4.8.5 Error Dynamics with Adaptive Controller

The error dynamics for the tracking error and the parameter estimation errors are given by:

$$\dot{e_1} = e_2$$

$$\dot{e_2} = -ae_1 - be_2 - (k_1(t) - a^*)y_1 - (k_2(t) - b^*)y_2$$

$$\dot{k_1} = e_1 y_1$$

$$\dot{k_2} = e_1 y_2$$

where $\tilde{k_1}(t) = k_1(t) - a^*$ and $\tilde{k_2}(t) = k_2(t) - b^*$ represent the estimation errors for the parameters.

4.8.6 Lyapunov Analysis for Stability

We define the Lyapunov function candidate as:

$$V = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}\tilde{k_1}^2 + \frac{1}{2}\tilde{k_2}^2$$

The time derivative of V is computed as:

$$\dot{V} = -ae_1^2 - be_2^2 - \tilde{k_1}e_1y_1 - \tilde{k_2}e_1y_2 + \tilde{k_1}e_1y_1 + \tilde{k_2}e_1y_2$$

This simplifies to:

$$\dot{V} = -ae_1^2 - be_2^2$$

Thus, $\dot{V}$ is negative semi-definite, ensuring that V does not increase over time. This guarantees that the errors $e_1(t)$, $e_2(t)$, $\tilde{k_1}(t)$, and $\tilde{k_2}(t)$ are bounded, and the tracking error $e(t) \to 0$ as $t \to \infty$.

4.8.7 Stability and Convergence

From the Lyapunov analysis, we conclude that the system will be stable, and the tracking error will converge to zero. Furthermore, the parameter estimates $k_1(t)$ and $k_2(t)$ will remain bounded, and under certain conditions, they will asymptotically approach the true values a^* and b^*, respectively.

4.9 Backstepping Methodology

Suppose a stabilizing feedback $u = \alpha(X)$, $\alpha(0) = 0$, is available for:

$$\dot{X} = F(X) + G(X)u, \quad X \in \mathbb{R}^n, \quad u \in \mathbb{R}, \quad F(0) = 0, \tag{4.34}$$

along with a Lyapunov function V such that

$$\frac{\partial V}{\partial X}\left(F(X) + G(X)\alpha(X)\right) \leq -W(X) < 0 \quad \forall X \neq 0. \tag{4.35}$$

Can we modify $\alpha(X)$ to stabilize the augmented system below?

$$\dot{X} = F(X) + G(X)x \tag{4.36}$$

$$\dot{x} = u. \tag{4.37}$$

4.9.1 Backstepping Procedure

The backstepping technique is a recursive design method for constructing stabilizing control laws for a special class of nonlinear systems known as **strict-feedback systems**. It works by building Lyapunov functions in a step-by-step manner, introducing auxiliary (virtual) controls and error coordinates at each step.

Define the error variable $z = x - \alpha(X)$ and change variables $(X, x) \to (X, z)$:

$$\dot{X} = F(X) + G(X)\alpha(X) + G(X)z \tag{4.38}$$

$$\dot{z} = u - \dot{\alpha}(X, z) \tag{4.39}$$

where $\dot{\alpha}(X, z) = \frac{\partial \alpha}{\partial X}\left(F(X) + G(X)\alpha(X) + G(X)z\right)$.
Take the new Lyapunov function:

$$V^{+}(X, z) = V(X) + \frac{1}{2}z^2. \tag{4.40}$$

$$\dot{V}^{+} = \underbrace{\frac{\partial V}{\partial X}\left(F(X) + G(X)\alpha(X)\right)}_{\leq -W(X)} + \frac{\partial V}{\partial X}G(X)z + z(u - \dot{\alpha}) \tag{4.41}$$

$$= \underbrace{\frac{\partial V}{\partial X}\left(F(X) + G(X)\alpha(X)\right)}_{\leq -W(X)} + z\left(u - \dot{\alpha} + \frac{\partial V}{\partial X}G(X)\right) \tag{4.42}$$

Let: $u = \dot{\alpha} - \frac{\partial V}{\partial X}G(X) - kz, \quad k > 0.$
Then, $\dot{V}^{+} \leq -W(X) - kz^2 \Rightarrow (X, z) = 0$ is asymptotically stable.

4.10 Examples

4.10.1 Example 1: Pure Integrator Backstepping

$$\dot{x}_1 = x_1^2 + x_2 \tag{4.43}$$

$$\dot{x}_2 = u \tag{4.44}$$

Treat x_2 as "virtual" control input for the x_1-subsystem:

$$\alpha(x_1) = -k_1 x_1 - x_1^2, \quad k_1 > 0 \tag{4.45}$$

$$V_1(x_1) = \frac{1}{2}x_1^2 \tag{4.46}$$

Apply backstepping:

$$z_2 = x_2 - \alpha(x_1) = x_2 + k_1 x_1 + x_1^2 \tag{4.47}$$

$$\dot{z}_2 = u - \dot{\alpha} \tag{4.48}$$

The control law is:

$$u = \dot{\alpha} - \frac{\partial V_1}{\partial x_1} - k_2 z_2, \quad k_2 > 0 \tag{4.49}$$

$$= \underbrace{-(k_1 + 2x_1)(x_1^2 + x_2)}_{\dot{\alpha}} - \underbrace{x_1}_{\frac{\partial V_1}{\partial x_1}} - k_2 \underbrace{(x_2 + k_1 x_1 + x_1^2)}_{z_2} \tag{4.50}$$

Generalization

We can generalize backstepping to systems of the form:

$$\dot{X} = F(X) + G(X)x \tag{4.51}$$

$$\dot{x} = f(X,x) + g(X,x)u \tag{4.52}$$

where $X \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $g(X,x) \neq 0$ for all $(X,x) \in \mathbb{R}^{n+1}$.
With the preliminary feedback:

$$u = \frac{1}{g(X,x)}(-f(X,x) + v) \tag{4.53}$$

the x-subsystem becomes a pure integrator: $\dot{x} = v$.
Substituting the backstepping control law:

$$v = \dot{\alpha} - \frac{\partial V}{\partial X}G(X) - kz, \quad z \triangleq x - \alpha(X), \quad k > 0 \tag{4.54}$$

into the preliminary feedback, we get:

$$u = \frac{1}{g(X,x)}\left(-f(X,x) + \dot{\alpha} - \frac{\partial V}{\partial X}G(X) - kz\right) \tag{4.55}$$

4.10.2 Recursive Backstepping for Strict Feedback Systems

Backstepping can be applied recursively to systems of the form:

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \tag{4.56}$$

$$\dot{x}_2 = f_2(x_1,x_2) + g_2(x_1,x_2)x_3 \tag{4.57}$$

$$\dot{x}_3 = f_3(x_1,x_2,x_3) + g_3(x_1,x_2,x_3)x_4 \tag{4.58}$$

$$\vdots \tag{4.59}$$

$$\dot{x}_n = f_n(x) + g_n(x)u \tag{4.60}$$

where $g_i(x_1,\ldots,x_i) \neq 0$ for all $x \in \mathbb{R}^n$, $i = 1,2,\ldots,n$.

Systems of this form are called "strict feedback systems."

4.10.3 Example 2: Non-stabilizable System

$$\dot{x}_1 = (x_1x_2 - 1)x_1^3 + (x_1x_2 + x_3^2 - 1)x_1 \tag{4.61}$$

$$\dot{x}_2 = x_3 \tag{4.62}$$

$$\dot{x}_3 = u \tag{4.63}$$

This system is not in strict feedback form because x_3 appears too soon. In fact, this system is not globally stabilizable because the set $x_1x_2 \geq 2$ is positively invariant regardless of u.

Consider the normal vector $n(x) = [x_2, x_1, 0]^T$ to the surface $x_1x_2 = 2$. Computing the dot product:

$$n(x) \cdot f(x,u) = [(x_1x_2 - 1)x_1^3 + (x_1x_2 + x_3^2 - 1)x_1]x_2 + x_3x_1 \tag{4.64}$$

Substituting $x_1 x_2 = 2$:

$$= (x_1^3 + (1 + x_3^2)x_1)x_2 + x_3 x_1 \tag{4.65}$$

$$= (x_1^2 + (1 + x_3^2))x_1 x_2 + x_3 x_1 \tag{4.66}$$

$$= 2x_1^2 + 2(1 + x_3^2) + x_3 x_1 \tag{4.67}$$

$$= 2x_1^2 + x_3 x_1 + 2x_3^2 + 2 > 0 \tag{4.68}$$

Since this dot product is always positive, trajectories can only cross the surface $x_1 x_2 = 2$ from below, making the region $x_1 x_2 \geq 2$ positively invariant.

4.10.4 Example 3: Relaxed Conditions

The condition $g_i(x_1, \ldots, x_i) \neq 0$ can be relaxed in some cases:

$$\dot{x}_1 = x_1^2 x_2 \tag{4.69}$$

$$\dot{x}_2 = u \tag{4.70}$$

Treat x_2 as virtual control and let $\alpha_1(x_1) = -x_1$ which stabilizes the x_1-subsystem, as verified with Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$.

Then $z_2 := x_2 - \alpha_1(x_1)$ satisfies $\dot{z}_2 = u - \dot{\alpha}_1$, and

$$u = \dot{\alpha}_1 - \frac{\partial V_1}{\partial x_1} x_1^2 - k_2 z_2 = -x_1^2 x_2 - x_1^3 - k_2(x_2 + x_1) \tag{4.71}$$

achieves global asymptotic stability:

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \Rightarrow \dot{V} = -x_1^4 - k_2 z_2^2 \tag{4.72}$$

Note that we can't conclude exponential stability due to the quartic term x_1^4 above. In fact, the linearization of the closed-loop system proves the lack of exponential stability:

$$\begin{bmatrix} 0 & 0 \\ 0 & -k_2 \end{bmatrix} \Rightarrow \lambda_{1,2} = 0, -k_2 \tag{4.73}$$

4.11 Design Example: Active Suspension System

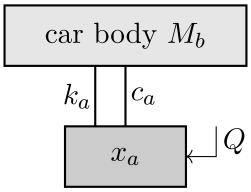


Figure 4.4: Active suspension system

The system dynamics are:

$$M_b \ddot{x}_s = -k_a(x_s - x_a) - c_a(\dot{x}_s - \dot{x}_a) \tag{4.74}$$

$$\dot{x}_a = \frac{1}{A} Q \quad \text{(A: effective piston surface)} \tag{4.75}$$

$$\dot{Q} = -c_f Q + k_f u \quad \text{(u: current applied to the solenoid valve)} \tag{4.76}$$

Define state variables: $x_1 = x_s$, $x_2 = \dot{x}_s$, $x_3 = x_a$, $x_4 = Q$:

$$\dot{x}_1 = x_2 \tag{4.77}$$

$$\dot{x}_2 = -\frac{k_a}{M_b}(x_1 - x_3) - \frac{c_a}{M_b}(x_2 - \frac{1}{A}x_4) \tag{4.78}$$

$$\dot{x}_3 = \frac{1}{A}x_4 \tag{4.79}$$

$$\dot{x}_4 = -c_f x_4 + k_f u \tag{4.80}$$

This system is not in strict feedback form due to the x_4 term in $\dot{x}_2$. To overcome this problem define:

$$\bar{x}_3 \triangleq \frac{k_a}{M_b}x_3 + \frac{c_a}{M_b A}x_4 \tag{4.81}$$

$$\xi \triangleq x_3 \tag{4.82}$$

and change variables to $(x_1, x_2, \bar{x}_3, \xi)$:

$$\dot{x}_1 = x_2 \tag{4.83}$$

$$\dot{x}_2 = -\frac{k_a}{M_b}x_1 - \frac{c_a}{M_b}x_2 + \bar{x}_3 \tag{4.84}$$

$$\dot{\bar{x}}_3 = \frac{k_a - c_a c_f}{M_b A}x_4 + \frac{c_a k_f}{M_b A}u \tag{4.85}$$

Two steps of backstepping starting with the virtual control law:

$$\alpha_1(x_1) = -c_1 x_1 - k_1 x_1^3 \tag{4.86}$$

will stabilize the $(x_1, x_2, \bar{x}_3)$ subsystem. The stiff nonlinearity $k_1 x_1^3$ prevents large excursions of x_1.

For the full $(x_1, x_2, \bar{x}_3, \xi)$ system, the ξ-subsystem is:

$$\dot{\xi} = -\frac{k_a}{M_b A}\xi + \frac{1}{A}\bar{x}_3 \tag{4.87}$$

The ξ-subsystem is an asymptotically stable linear system driven by $\bar{x}_3$; therefore the full system is stabilized.