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Thème

**Modélisation d'un système automatique
à base d'un pendule inversé**

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

Dedication

Shams of Tabriz said "when you step into the zone of love, language as we know it becomes obsolete" my lovers no matter what I say I will never thank you enough. I have the honor to dedicate this work to

My parents FRIH Habib and ZIANE Fatiha, my dears, I owe you everything because you made me as I am, without you I would not have achieved anything.

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Abstract

Our main goal in this thesis is to simulate different control laws and strategies. This work is dedicated mostly to the theoretical study of an inverted pendulum, we explained the dynamics and state representation of one and two inverted pendulums and generalise the equations of motion until we reach the n -link pendulums. We also designed a controller for the swing-up through the study of energy and designed our LQR controller for stabilization. Finally, we designed a controller for the two inverted pendulums and talked about some control laws. As result, we simulated the control law that we talked about and compared between them.

Key words inverted pendulum, n -link inverted pendulum, LQR, Pole placement, swing-up.

ملخص

هدفنا الرئيسي في هذه الأطروحة هو محاكاة قوانين واستراتيجيات التحكم المختلفة. هذا العمل المخصص في الغالب للدراسة النظرية للنواس المقلوب ، و شرح الديناميكيات وتمثيل الحالة لنواس واحد واثنين من النواس المقلوبة و من ثم عممنا معادلات الحركة حتى صلنا إلى النواس $n-link$. قمنا أيضًا بتصميم وحدة تحكم للتأرجح من خلال دراسة الطاقة وتصميم وحدة التحكم LQR الخاصة بنا لتحقيق الاستقرار. أخيرًا ، صممنا وحدة تحكم للنواسين المقلوبين وتحدثنا عن بعض قوانين التحكم. نتيجة لذلك ، قمنا بمحاكاة قانون التحكم الذي تحدثنا عنه وقارنا بينهما. **كلمات مفتاحية** نواس مقلوب، نواس مقلوب، متحكم تربيعي خطي، تنسيب الاقطاب، التأرجح.

Résumé

Notre objectif principal dans cette thèse est de simuler différentes lois et stratégies de commande. Ce travail est dédié principalement à l'étude théorique du pendule inversé, nous avons expliqué la dynamique et la représentation de l'état d'un et deux pendules inversés et généralisé les équations de mouvement jusqu'à ce que nous atteignons les pendules à n liens, nous avons aussi conçu un contrôleur pour le swing-up à travers l'étude de l'énergie et un contrôleur LQR pour la stabilisation. Finalement, nous avons conçu un contrôleur pour les deux pendules inversés, et parlé de certaines lois de contrôle. Comme résultat, nous avons simulé la loi de commande dont nous avons parlé et essayer de les comparer.

mots clés pendule inversé , pendule inversé n -lien, LQR, placement de pole, balancer.

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General introduction

Almost all of us have already tried the experience of vertically stabilizing a stick on our index finger. In order to keep this stick in balance, we must move our index so as to permanently prevent it from tipping over. It's exactly the same challenge offered by the inverted pendulum system, but the motion of the inverted pendulum system consists of translational movement on limited rail and rotational movement. It will be necessary to define specific strategies to ensure automated maintenance of the inverted pendulum. And of course, the performances obtained thanks to an automated system are far superior to those which would be obtained by man.

To define the specific strategies that we talked about we need to understand the mathematical model [1].

The strongly non-linear and unstable dynamics of this physical system have made it a very interesting and widely studied system in the automation community. As well, in the scientific field and education, automatic control often uses specific case studies, which are representative of large classes of applications. Moreover, with experience, the knowledge of these cases has been refined and they now provide an ideal basis for validly comparing the advantages and disadvantages of different approaches.

The inverted pendulum is one of those typical cases that occupies an important place in the industry as a transport tool, which has been the subject of great discussion throughout the last years. This explains the existence of several aspects and forms of inverted pendulums [2]. We can find various types of inverted pendulums such as the simple inverted pendulum, the double

inverted pendulum, the Furuta inverted pendulum, etc.

Our system belongs to the class of fast single input, multiple-output (SIMO), it is an under-actuated system and satisfies a set of complicated characteristics, such as fourth-order highly non-linear dynamics, open-loop instability, state coupling, and non-minimum-phase (NMP) behavior which affects stability margins and robustness, provides many challenging problems to standard and modern control techniques [1].

In the context of the inverted pendulum on a cart system stabilization, moving the cart from an initial position to a final destination while keeping the pendulum erected in the upright position has been extensively studied in the past, and many output-feedback and (static and dynamic) state-feedback control techniques have been developed to solve it [3].

Nearly all work on pendulum control concentrate on two problems: pendulums swing up control design and stabilization of the inverted pendulums. There are different types of control methods that have been applied to inverted pendulum systems, such as PID controller, a combination of feedforward and feedback control to solve the swing-up of an inverted pendulum problem was proposed in (Rubi et al., 2002). (Graichen et al., 2007) , LQR controller, A side-stepping control, and there is more even the fuzzy logic was applied.

This work is divided into 3 chapters, in the first one we extracted the equations of motion of one inverted pendulum using language's equation, and after we turn these equations of motion into state space representation. Next, we extracted the equations of motion of two inverted pendulums and wrote them in matrix form eventually from those sections and after some research, we managed to get the equations of motions for the n-link inverted pendulum. In the second chapter, we divided the problem of controlling an inverted pendulum onto two problems the swing-up control and the stabilisation control. started with the swing-up control we did a study based on the energy of the system after we got our equilibrium points and then we linearised our system around those equilibrium points next we did the stabilisation control of one inverted pendulum we tasted its stability by Lyapunov function. In the next section of this chapter, we tried to design an LQR controller for two inverted pendulums after we talked about different control strategies to stabilise the pendulum. In the last chapter by using MATLAB we managed to get the simulation of one

inverted pendulum on a cart, and by using the ODE45 function we solve the dynamics equations and then we simulate it in an open loop with and without friction after we simulate our system with a different control method.

Mathematical modeling of an inverted pendulum

1.1 Introduction

The inverted pendulum on a cart system is a multi-variable, coupled, non-linear, and unstable system. there are many parasitic effects such as friction, and the rotational movement of an electric motor is transformed into the movement of translation of the carriage using a pulley and a toothed belt. The movement of the carriage in one direction or the other ensures by reaction the vertical balance of the pendulum arm. The mathematical model of a dynamic is defined as a set of equations that represents the dynamics of the system accurately. It is the best and the most important tool to describe the physical system, so we can design a controller and simulate it, we will extract the potential and the kinetic energies, and by applying the Lagrange method we will deduce the mathematical model. Finally, we can represent it as state space form, or transfer function form. In this chapter, we will try to deduct the mathematical model of one and two inverted pendulums on a cart, and we will deduct just the equations of motion of n-link inverted pendulums.

1.2 One inverted pendulum mathematical model

1.2.1 Describing the system

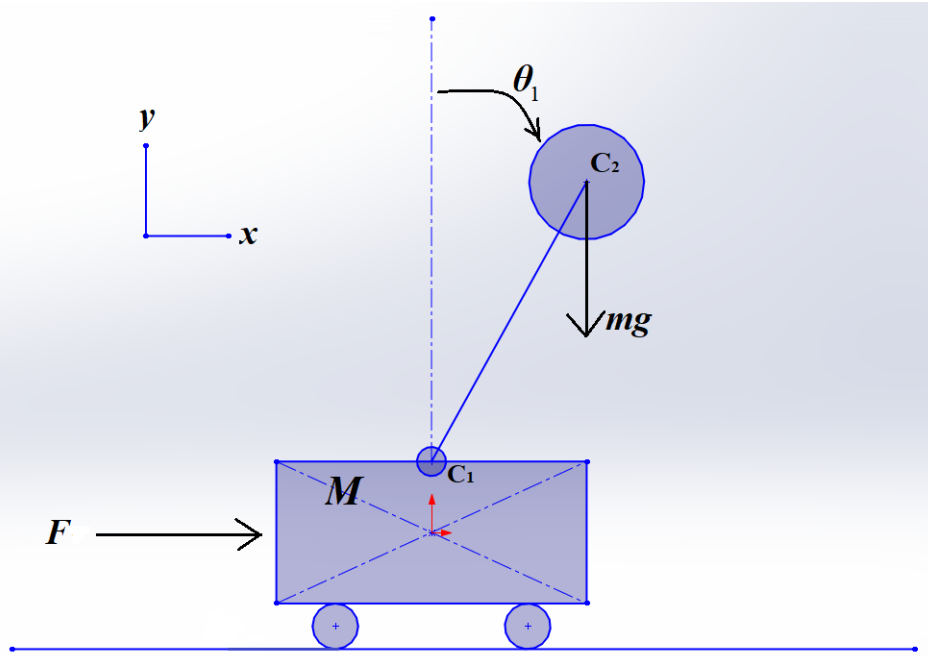


Figure 1.1: 1 link inverted pendulum

This mechanical system consists of a carriage of mass M which can move horizontally and freely on a guide rail by supporting a rod of mass m_1 and free in rotation around a pivot. The DC motor will apply a force F (the control input it's the force F we can describe it as $u = u(t)$) (see fig 1.1) that causes the cart to move so it has two degrees of freedom whose coordinates generalized are respectively x for the horizontal movement of the carriage and θ for the rotation of the pendulum, as the pendulum rotate that involved the existence of moment of inertia I [4], and the positive direction of x is the right direction in meters and that of the angle is the clockwise direction in radians. There are many different modelling strategies, one of them by determining the equations of motion of the pendulum by applying Lagrange's formalism. Lagrange's equations are one of the best-known methods for solving dynamic problems with constraints. The Lagrangian L is defined as $L = K - P$, where K is the kinetic energy and P is the potential energy of the system in question.

The length of the pendulum is denoted by $2l$ so the gravity center is in distance equal to l .

We denote by $\theta_1 = \theta_1(t)$ the deviation of the pendulum from the upright position at time $t \in \mathfrak{R}$ as depicted in the image above. By $x = x(t)$ we denote the horizontal position of the cart and we assume that the cart cannot move vertically [5].

The derivatives with respect to time are denoted by

$$\frac{d}{dt}x(t) = \dot{x}, \text{ and } \frac{d}{dt}\theta_1$$

The goal is to stabilize the pendulum in an upright position above the cart by only applying forces to the cart itself.

We assume that external disturbances w_0, w_1 , act as forces on x and θ_1 think of these external forces as wind or some human pushing the rods. We introduce the damping coefficients d_0, d_1 and consider the friction/damping force of the cart to be $-d_0\dot{x}$ while the friction/damping forces in the joints are assumed to be $-d_1\dot{\theta}_1$ and the positions of the masses M and m_1 are given by

Coordinates of the system :

The cart

$$c_0 = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

the pendulum:

$$c_1 = \begin{bmatrix} Xm \\ Ym \end{bmatrix} = \begin{bmatrix} x + l \sin \theta \\ l \cos \theta \end{bmatrix}$$

1.2.2 The potential energy

The cart only moves on a horizontal rail, so its potential energy is zero, therefore only the moving pendulum has potential energy, so the potential energy in the pendulum can be given as

$$\begin{aligned} P &= mgh \\ &= m_1gl \cos \theta_1 \end{aligned}$$

1.2.3 The kinetic energy

We know that the pendulum moves in horizontal rail and rotates on its own axe of rotation, this rotational movement causes rotational kinetic energy we can describe it as

$$K_{rot} = \frac{1}{2}I\omega^2$$

with ω is the angular velocity. So the kinetic energy can be given as

$$\begin{aligned} K &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2} \{M \|\dot{c}_0\|^2 + m_1 \|\dot{c}_1\|^2\} + \frac{1}{2}I\dot{\theta}_1^2 \\ K_1 &= \frac{1}{2}M\dot{x}^2 \\ K_2 &= \frac{1}{2}m_1(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}_1^2 \\ K &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}_1^2 \\ &= \frac{1}{2} \left\{ M\dot{x}^2 + m_1 \left[(\dot{x} + l\dot{\theta}_1 \cos \theta_1)^2 + (l\dot{\theta}_1 \sin \theta_1)^2 \right] \right\} + \frac{1}{2}I\dot{\theta}_1^2 \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_1(l\dot{\theta}_1 \cos \theta_1)^2 \\ &\quad + m_1\dot{x}l\dot{\theta}_1 \cos \theta_1 + \frac{1}{2}m_1(l\dot{\theta}_1 \sin \theta_1)^2 + \frac{1}{2}I\dot{\theta}_1^2 \end{aligned}$$

1.2.4 Lagrange's equation

$$L = K - P$$

$$L = \frac{1}{2}(M + m_1)\dot{x}^2 + \frac{1}{2}m_1l^2\dot{\theta}_1^2 + m_1l\dot{\theta}_1 \cos \theta_1 \dot{x} + \frac{1}{2}I\dot{\theta}_1^2 - m_1gl \cos \theta_1$$

1.2.5 The equations of motion

$$\begin{aligned}
 u + w_0 - d_0\dot{x} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \\
 &= \frac{d}{dt}\left\{M\dot{x} + m_1(\dot{x} + l\dot{\theta}_1 \cos \theta_1)\right\} - \{0\} \\
 &= (M + m_1)\ddot{x} + m_1l\ddot{\theta}_1 \cos \theta_1 - m_1l\dot{\theta}_1^2 \sin \theta_1 \\
 w_1 - d_1\dot{\theta}_1 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} \\
 &= \frac{d}{dt}\left\{m_1l^2\dot{\theta}_1 + m_1l\dot{x} \cos \theta_1 + I\dot{\theta}_1\right\} + \left\{-lm_1\dot{x} \sin \theta_1 \dot{\theta}_1 + m_1gl \sin \theta_1\right\} \\
 &= m_1l\ddot{x} \cos \theta_1 + l^2m_1\ddot{\theta}_1 + I\ddot{\theta}_1 - gm_1l \sin \theta_1
 \end{aligned}$$

as the pendulum can rotate without a big resistance and it is easy to accelerate it so we can neglect the moment of inertia, therefore the second equation of motion can be given as

$$w_1 - d_1\dot{\theta}_1 = m_1l\ddot{x} \cos \theta_1 + l^2m_1\ddot{\theta}_1 - gm_1l \sin \theta_1$$

1.2.6 State space representation

Our system can be presented in the state space form [6], and we consider disturbance $w = 0$ and the inertia moment $I = 0$:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + G(q) = F \tag{1.1}$$

where

$$\begin{aligned}
 q &= \begin{bmatrix} x \\ \theta_1 \end{bmatrix} \\
 M(q) &= \begin{bmatrix} M + m_1 & m_1l \cos \theta_1 \\ m_1l \cos \theta_1 & m_1l^2 \end{bmatrix} \tag{1.2}
 \end{aligned}$$

$$C(q, \dot{q}) = \begin{bmatrix} 0 & -m_1 l \sin \theta_1 \dot{\theta}_1 \\ 0 & 0 \end{bmatrix} \quad (1.3)$$

$$D = \begin{bmatrix} d_0 & 0 \\ 0 & d_1 \end{bmatrix} \quad (1.4)$$

$$G(q) = \begin{bmatrix} 0 \\ -m_1 g l \sin \theta_1 \end{bmatrix}, F = \begin{bmatrix} u \\ 0 \end{bmatrix} \quad (1.5)$$

The matrix $M(q)$ is a symmetric inertia matrix with a determinant equal to

$$\begin{aligned} \det(M(q)) &= (M + m_1)m_1 l^2 - (m_1 l \cos \theta_1)^2 \\ &= m_1 l^2 (M + m_1 \sin^2 \theta_1) > 0 \end{aligned} \quad (1.6)$$

that shows it is positive definite for all q . $C(q, \dot{q})$ represents centrifugal and Coriolis forces. In term D there are viscous damping coefficients while $G(q)$ accounts for gravitational forces and is given as the derivative concerning q of the potential energy $P(q)$. Moreover, through equations 1.11, 1.3 is defined

$$\dot{M}(q) - 2C(q, \dot{q}) = \begin{bmatrix} 0 & m_1 l \sin \theta_1 \dot{\theta}_1 \\ -m_1 l \sin \theta_1 \dot{\theta} & 0 \end{bmatrix} \quad (1.7)$$

that is a skew-symmetric matrix, that follows

$$z^T (\dot{M}(q) - 2C(q, \dot{q})) z = 0 \quad (1.8)$$

$\forall z \in \mathfrak{R}^2$

Using the system 1.1 the term \ddot{q} can be obtained from

$$\ddot{q} = M(q)^{-1} (-C(q, \dot{q})\dot{q} - D\dot{q} - G(q) + F)$$

The term M^{-1} can be attained using 1.11, 1.3, 1.6 and is defined as

$$M^{-1} = \frac{1}{\det(M)} \begin{bmatrix} m_1 l^2 & -m_1 l \cos \theta_1 \\ -m_1 l \cos \theta_1 & M + m_1 \end{bmatrix} \quad (1.9)$$

Thus, the nonlinear state space model is reached:

$$\begin{aligned} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \end{bmatrix} &= \frac{1}{\det(M)} \begin{bmatrix} m_1 l^2 & -m_1 l \cos \theta_1 \\ -m_1 l \cos \theta_1 & M + m_1 \end{bmatrix} \left[(-C) \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \end{bmatrix} + (-D) \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \end{bmatrix} + (-G) + F \right] \\ &= \frac{1}{\det(M)} \begin{bmatrix} m_1 l^2 & -m_1 l \cos \theta_1 \\ -m_1 l \cos \theta_1 & M + m_1 \end{bmatrix} \left\{ \begin{bmatrix} 0 & m_1 l \sin \theta_1 \dot{\theta}_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} -d_0 & 0 \\ 0 & -d_1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ m_1 g l \sin \theta_1 \end{bmatrix} + \begin{bmatrix} u \\ 0 \end{bmatrix} \right\} \\ &= \frac{1}{\det(M)} \begin{bmatrix} m_1 l^2 (m_1 l \sin \theta_1 \dot{\theta}_1^2 - d_0 \dot{x} + u) - m_1 l \cos \theta_1 (-d_1 \dot{\theta}_1 + m_1 g l \sin \theta_1) \\ -m_1 l \cos \theta_1 (m_1 l \sin \theta_1 \dot{\theta}_1^2 - d_0 \dot{x} + u) + (M + m_1) (-d_1 \dot{x} + m_1 g l \sin \theta_1) \end{bmatrix} \\ &= \frac{1}{\det(M)} \begin{bmatrix} m_1^2 l^3 \sin \theta_1 \dot{\theta}_1^2 - m_1 l^2 d_0 \dot{x} + m_1 l \cos \theta_1 d_1 \dot{\theta}_1 - m_1^2 g l^2 \cos \theta_1 \sin \theta_1 \\ -m_1^2 l^2 \cos \theta_1 \sin \theta_1 \dot{\theta}_1^2 + m_1 l \cos \theta_1 d_0 \dot{x} - (M + m_1) d_1 \dot{\theta}_1 + (M + m_1) m_1 g l \sin \theta_1 \end{bmatrix} \\ &\quad + \begin{bmatrix} m_1 l^2 u \\ -m_1 l \cos \theta_1 u \end{bmatrix} \\ \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \end{bmatrix} &= \frac{1}{\det(M)} \left(\begin{bmatrix} 0 & m_1^2 l^3 \dot{\theta}_1 \sin \theta_1 \\ 0 & -m_1^2 l^2 \dot{\theta}_1 \sin \theta_1 \cos \theta_1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} -m_1^2 l^2 g \sin \theta_1 \cos \theta_1 \\ (M + m_1) m_1 g l \sin \theta_1 \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} -m_1 l^2 d_0 & m_1 l \cos \theta_1 d_1 \\ m_1 l \cos \theta_1 d_0 & -(M + m_1) d_1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} m_1 l^2 u \\ -m_1 l \cos \theta_1 u \end{bmatrix} \right) \quad (1.10) \end{aligned}$$

to get out non-linear state representation we need to find our state variables as shown :

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ x_3 = \theta_1 \\ x_4 = \dot{\theta}_1 \end{cases}$$

which defines the state equations for \ddot{x} and $\ddot{\theta}_1$, we get this non-linear equations :

$$\begin{cases} \dot{x}_1 = \dot{x} = x_2 = f_1(x, \theta_1, u) \\ \dot{x}_2 = \ddot{x} = f_2(x, \theta_1, u) \\ \dot{x}_3 = x_4 = \dot{\theta}_1 = f_3(x, \theta_1, u) \\ \dot{x}_4 = \ddot{\theta}_1 = f_4(x, \theta_1, u) \end{cases}$$

Thus we have

$$\begin{aligned} \ddot{x} &= \frac{m_1^2 l^3 \sin \theta_1 \dot{\theta}_1^2 - m_1^2 l^2 g \sin \theta_1 \cos \theta_1 m_1 l^2 d_0 \dot{x} + m_1 l \cos \theta_1 d_1 \dot{\theta}_1 + m_1 l^2 u}{m_1 l^2 (M + m \sin^2 \theta_1)} \\ &= \frac{m_1 \sin \theta_1 (l \dot{\theta}_1^2 - g \cos \theta_1) - d_0 \dot{x} + \frac{d_1 \cos \theta_1 \dot{\theta}_1}{l}}{M + m_1 \sin^2 \theta_1} + \frac{u}{M + m_1 \sin^2 \theta_1} \end{aligned} \quad (1.11)$$

$$\begin{aligned} \ddot{\theta}_1 &= \frac{-m_1^2 l^2 \sin \theta_1 \cos \theta_1 \dot{\theta}_1^2 + m_1 l d_0 \cos \theta_1 \dot{x} - (M + m_1) d_1 \dot{\theta}_1 + (M + m_1) g m_1 l \sin \theta_1}{m_1 l^2 (M + m_1 \sin^2 \theta_1)} \\ &\quad + \frac{-m_1 l \cos \theta_1 u}{m_1 l^2 (M + m_1 \sin^2 \theta_1)} \end{aligned} \quad (1.12)$$

1.3 Equations of motions for double pendulum

The length of the first pendulum is denoted by $2l_1$ and from it, we have the gravity center of the first link is l_1 and the length of the second pendulum by l_2 so the gravity center of the second link is l_2 . The mass of the cart is denoted by M . The mass of the first pendulum m_1 , and The mass of the cart is denoted by M see fig 1.2. The mass of the first pendulum m_2 .

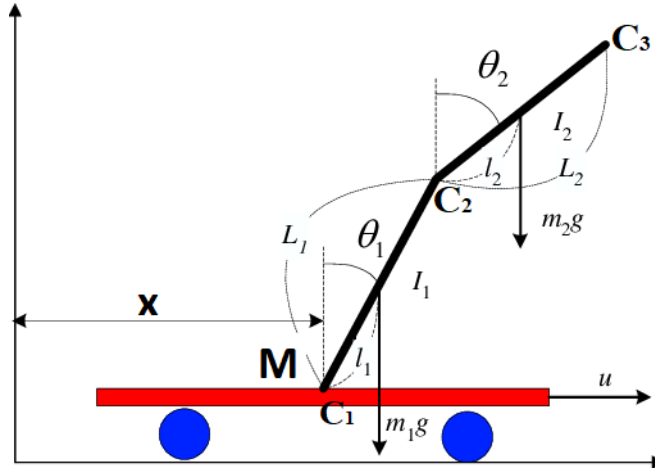


Figure 1.2: 2-link inverted pendulum

We denote by $\theta_1 = \theta_1(t)$ and $\theta_2 = \theta_2(t)$ the deviation of the rods from the upright position at time $t \in \mathfrak{R}$ as depicted in the image above. By $x = x(t)$ we denote the horizontal position of the cart and we assume that the cart cannot move vertically. The derivatives with respect to time are denoted by

$$\frac{d}{dt}x(t) = \dot{x}, \quad \frac{d}{dt}\theta_1(t) = \dot{\theta}_1, \quad \frac{d}{dt}\theta_2(t) = \dot{\theta}_2.$$

The control input $u = u(t)$ is thus the force that we can apply to the cart. Furthermore, we assume that external disturbances w_1, w_2, w_3 act as forces on x, θ_1, θ_2 , think of these external forces as wind or some human pushing the pendulum. The friction in the joints and the friction of the moving cart are modeled via a linear ansatz. We therefore introduce the damping coefficients d_0, d_1, d_2 and consider the friction or the damping force of the cart to be $-d_0\dot{x}$ while the friction or damping forces in the joints are assumed to be $-d_1\dot{\theta}_1$ and $-d_2\dot{\theta}_2$ we suppose that the moment of inertia $I = 0$ because it is too small [7].

The positions of the masses The positions of the masses $M, m_1,$ and m_2 are given by

$$c_0 = \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad c_1 = \begin{bmatrix} x + l_1 \sin \theta_1 \\ l_1 \cos \theta_1 \end{bmatrix}, \quad \text{and} \quad c_2 = \begin{bmatrix} x + l_1 \sin \theta_1 + l_2 \sin \theta_2 \\ l_1 \cos \theta_1 + l_2 \cos \theta_2 \end{bmatrix},$$

The kinetic energy of double inverted pendulum

respectively. Thus, the kinetic energy in the system is

$$\begin{aligned}
 K &= \frac{1}{2}[M|\dot{c}_0|^2 + m_1|\dot{c}_1|^2 + m_2|\dot{c}_2|^2] + \frac{1}{2}(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2) \\
 &= \frac{1}{2} \left\{ M\dot{x}^2 + m_1 \left[(\dot{x} + l_1\dot{\theta}_1 \cos \theta_1)^2 + (l_1\dot{\theta}_1 \sin \theta_1)^2 \right] \right. \\
 &\quad \left. + m_2 \left[(\dot{x} + l_1\dot{\theta}_1 \cos \theta_1 + l_2\dot{\theta}_2 \cos \theta_2)^2 \right. \right. \\
 &\quad \left. \left. + (l_1\dot{\theta}_1 \sin \theta_1 + l_2\dot{\theta}_2 \sin \theta_2)^2 \right] \right\} + \frac{1}{2}(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2) \\
 &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}(m_1l_1^2 + I_1)\dot{\theta}_1^2 + m_1l_1\dot{x}\dot{\theta}_1 \cos \theta_1 \\
 &\quad + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}(m_2l_2^2 + I_2)\dot{\theta}_2^2 \\
 &\quad + m_2l_1\dot{x}\dot{\theta}_1 \cos \theta_1 + m_2l_2\dot{x}\dot{\theta}_2 \cos \theta_2 \\
 &\quad + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)
 \end{aligned}$$

The potential energy of double inverted pendulum

It can be given as

$$P = g\{m_1l_1 \cos \theta_1 + m_2(l_1 \cos \theta_1 + l_2 \cos \theta_2)\}$$

Lagrange equation

$$\begin{aligned}
 L = & \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 \\
 & + \frac{1}{2}(m_1l_1^2 + m_2l_1^2 + I_1)\dot{\theta}_1^2 + \frac{1}{2}(m_2l_2^2 + I_2)\dot{\theta}_2^2 \\
 & + (m_1l_1 + m_2l_1) \cos(\theta_1)\dot{x}\dot{\theta}_1 \\
 & + m_2l_2 \cos \theta_2 \dot{x}\dot{\theta}_2 + m_2l_1l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \\
 & - (m_1l_1 + m_2l_1)g \cos \theta_1 - m_2l_2g \cos \theta_2
 \end{aligned}$$

by applying the principle of lagrangian mechanics we get the equations of motion, they are :

$$\begin{aligned}
 u + w_0 - d_0\dot{x} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \\
 &= \frac{d}{dt} \left\{ M\dot{x} + m_1 \left(\dot{x} + l_1\dot{\theta}_1 \cos \theta_1 \right) \right. \\
 &\quad \left. + m_2 \left(\dot{x} + l_1\dot{\theta}_1 \cos \theta_1 + l_2\dot{\theta}_2 \cos \theta_2 \right) \right\} \\
 &= (M + m_1 + m_2)\ddot{x} + l_1(m_1 + m_2)\ddot{\theta}_1 \cos \theta_1 \\
 &\quad + m_2l_2\ddot{\theta}_2 \cos \theta_2 - l_1(m_1 + m_2)\dot{\theta}_1^2 \sin \theta_1 - m_2l_2\dot{\theta}_2^2 \sin \theta_2
 \end{aligned}$$

$$\begin{aligned}
 w_1 - d_1\dot{\theta}_1 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} \\
 &= \left\{ l_1(m_1 + m_2)\dot{x}\dot{\theta}_1 \sin \theta_1 + l_1l_2m_2\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) - g(m_1 + m_2)l_1 \sin \theta_1 \right\} \\
 &\quad + \frac{d}{dt} \left\{ l_1(m_1 + m_2)\dot{x} \cos \theta_1 + l_1^2(m_1 + m_2 + I_1)\dot{\theta}_1 + l_1l_2m_2\dot{\theta}_2 \cos(\theta_1 - \theta_2) \right\} \\
 &= \left\{ l_1(m_1 + m_2)\dot{x}\dot{\theta}_1 \sin \theta_1 + l_1l_2m_2\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) - g(m_1 + m_2)l_1 \sin \theta_1 \right\} \\
 &\quad + \left\{ l_1(m_1 + m_2)\ddot{x} \cos \theta_1 + l_1^2(m_1 + m_2 + I_1)\ddot{\theta}_1 + l_1l_2m_2\ddot{\theta}_2 \cos(\theta_1 - \theta_2) \right. \\
 &\quad \left. - l_1(m_1 + m_2)\dot{x}\dot{\theta}_1 \sin \theta_1 - l_1l_2m_2\dot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= l_1(m_1 + m_2)\ddot{x} \cos \theta_1 + l_1^2(m_1 + m_2 + I_1)\ddot{\theta}_1 + l_1 l_2 m_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \\
 &\quad + l_1 l_2 m_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - g(m_1 + m_2)l_1 \sin \theta_1 \\
 w_2 - d_2 \dot{\theta}_2 &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} \\
 &= \left\{ -l_2 m_2 g \sin \theta_2 + l_2 m_2 \dot{x} \dot{\theta}_2 \sin \theta_2 - l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \right\} \\
 &\quad + \frac{d}{dt} \left\{ (l_2^2 m_2 + I_2) \dot{\theta}_2 + l_2 m_2 \dot{x} \cos \theta_2 + l_1 l_2 m_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \right\} \\
 &= \left\{ -l_2 m_2 g \sin \theta_2 + l_2 m_2 \dot{x} \dot{\theta}_2 \sin \theta_2 - l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \right\} \\
 &\quad + \left\{ (l_2^2 m_2 + I_2) \ddot{\theta}_2 + l_2 m_2 \ddot{x} \cos \theta_2 + l_1 l_2 m_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) \right. \\
 &\quad \left. - l_2 m_2 \dot{x} \dot{\theta}_2 \sin \theta_2 - l_1 l_2 m_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \right\} \\
 &= l_2 m_2 \ddot{x} \cos \theta_2 + l_1 l_2 m_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) \\
 &\quad + (l_2^2 m_2 + I_2) \ddot{\theta}_2 - l_1 l_2 m_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - l_2 m_2 g \sin \theta_2
 \end{aligned}$$

let y be

$$y = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

here also we consider the disturbance force equal to zero, we can describe our equations in matrix form

$$M(y)\ddot{y} = f(y, \dot{y}, u). \tag{1.13}$$

where

$$\underbrace{\begin{bmatrix} M + m_1 + m_2 & l_1(m_1 + m_2) \cos \theta_1 & m_2 l_2 \cos \theta_2 \\ l_1(m_1 + m_2) \cos \theta_1 & l_1^2(m_1 + m_2) & l_1 l_2 m_2 \cos(\theta_1 - \theta_2) \\ l_2 m_2 \cos \theta_2 & l_1 l_2 m_2 \cos(\theta_1 - \theta_2) & l_2^2 m_2 \end{bmatrix}}_{M_2(y)} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\ddot{y}}$$

$$= \underbrace{\begin{bmatrix} l_1(m_1 + m_2)\dot{\theta}_1^2 \sin \theta_1 + m_2 l_2 \dot{\theta}_2^2 \sin \theta_2 \\ -l_1 l_2 m_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2)l_1 \sin \theta_1 \\ l_1 l_2 m_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g l_2 m_2 \sin \theta_2 \end{bmatrix}}_{f(y, \dot{y}, u)} - \begin{bmatrix} d_0 \dot{x} \\ d_1 \dot{\theta}_1 \\ d_2 \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}$$

we calculate the determinant of $M_2(y)$ gives

$$\det M_2(y) = l_1^2 l_2^2 m_2 (M m_1 + m_1^2 \sin^2 \theta_1 + m_1 m_2 \sin^2 \theta_1 + M m_2 \sin^2(\theta_1 - \theta - 2)) > 0$$

as the $\det M_2(y) > 0$ we conclude that $M_2(y)$ is invertible for all $y \in \mathfrak{R}^3$, Thus we can rewrite 1.13 as

$$\ddot{y} = M_2^{-1}(y) f(y, \dot{y}, u)$$

with

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

So

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \underbrace{M_2^{-1}(y) f(y, \dot{y}, u)}_{F(x, u)} \end{bmatrix}$$

$$\dot{x} = F(x, u)$$

1.4 Equations of motion for N-inverted pendulum on a cart

By using the Lagrange method as we did in the previous sections, we will derive the equations of motions of the n-link inverted pendulum on a cart. Let's define the coordinate : x is the position of the cart and M is the mass of the cart, θ_i , $i = 1, \dots, n$, angle of i -th pendulum, with respect to the vertical axis. The characteristic of each pendulum is five physical parameters: the pendulum length l_i , the pendulum mass m_i , the relative position a_i of center of gravity, i.e. $|A_{i-1}T_i| = a_i l_i$, where T_i is the center of gravity of the i -th pendulum, A_i denotes the i -th

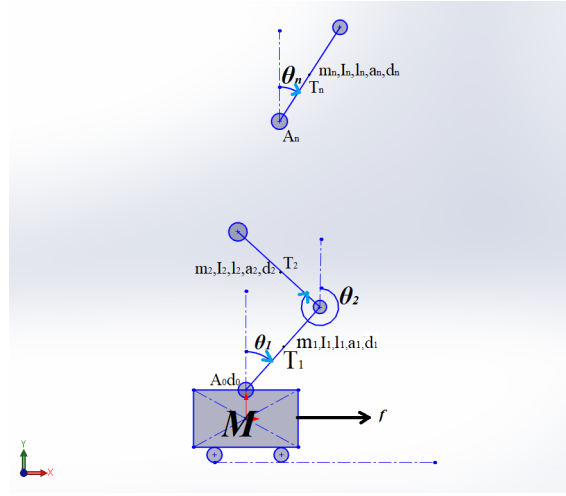


Figure 1.3: n-link inverted pendulum

revolute joint (see Fig 1.3), I_i denotes the moment of inertia about the axis through the center of gravity, and d_i denotes the coefficient of friction in the i -th joint, thus the total number of these parameters is $5n$. Unfortunately, these parameters can not be directly identifiable. In this section, we show, that for arbitrary n , we can introduce the vector of the standard dynamic parameters δ , which are directly identifiable [8].

Consider the notations introduced in Fig. 1.3, and denote the x-coordinate of the point T_i as X_i , and its y-coordinate as Y_i , $i = 1, 2, \dots, n$. Now, it follows

$$X_i = \sum_{j=1}^i l_j \sin \theta_j + a_i l_i \sin \theta_i + x$$

$$Y_i = \sum_{j=1}^i l_j \cos \theta_j + a_i l_i \cos \theta_i$$

Potential energy of n-link inverted pendulum

For potential energy P and kinetic energy K of the n-link inverted pendulum we get

$$P = \sum_{k=1}^n m_k g Y_k$$

$$K = \sum_{k=1}^n \frac{1}{2} \left\{ m_k \left[\left(\frac{dX_k}{dt} \right)^2 + \left(\frac{dY_k}{dt} \right)^2 \right] \right\} + \frac{1}{2} M \left(\frac{dx}{dt} \right)^2 \quad (1.14)$$

Equation (1.14) leads to arbitrary n to

$$\begin{aligned} P_n &= \sum_{i=1}^n m_i g \left(\sum_{j=1}^{i-1} l_j \cos \theta_j + a_i l_i \cos \theta_i \right) \\ K_n &= \sum_{k=1}^n \left[\frac{1}{2} \sum_{i=1}^{k-1} m_k l_i^2 \dot{\theta}_i^2 + \sum_{j=1, j>i}^{k-1} \sum_{i=1}^{k-1} m_k l_i l_j \dot{\theta}_i \dot{\theta}_j \cos(\theta_i - \theta_j) \right. \\ &+ \sum_{j=1}^{k-1} m_k a_k l_j l_k \dot{\theta}_j \dot{\theta}_k \cos(\theta_j - \theta_k) + \sum_{j=1}^{k-1} m_k l_j \dot{\theta}_j \dot{x} \cos \theta_j + \frac{1}{2} m_k \dot{x}^2 \\ &\left. + \frac{1}{2} m_k a_k^2 l_k^2 \dot{\theta}_k^2 + m_k a_k l_k \dot{\theta}_k \dot{x} \cos \theta_k + \frac{1}{2} I_k \dot{\theta}_k \right] + \frac{1}{2} M \dot{x}^2 \end{aligned}$$

Proposition 1: For arbitrary n, the potential energy of the n link inverted pendulum on a cart can be expressed as

$$P_n = \sum_{i=1}^n v_i g \cos \theta_i$$

where

$$v_i = m_i a_i l_i + l_i \left(\sum_{j=i+1}^n m_j \right), i = 1, \dots, n.$$

kinetic energy of n-link inverted pendulum

Proposition 2: For arbitrary n, the kinetic energy of the n link inverted pendulum on a cart can be expressed as

$$\begin{aligned} K_n &= \frac{1}{2} \sum_{i=1}^n k_i \dot{\theta}_i^2 + \sum_{j=2, j>i}^n \sum_{i=1}^{n-1} \mu_{ij} \dot{\theta}_i \dot{\theta}_j \cos(\theta_i - \theta_j) \\ &+ \sum_{i=1}^n v_i \dot{\theta}_i \dot{x} \cos \theta_i + \frac{1}{2} \dot{x}^2 \sum_{i=0}^n m_i, \end{aligned}$$

where

$$k_i = I_i + m_i a_i^2 l_i^2 + l_i^2 \sum_{k=i+1}^n m_k, i = 1 \dots, n$$

and

$$\mu_{ij} = l_i(m_j a_j l_j + l_j \sum_{k=j+1}^n m_k), 2 \leq j \leq n, 1 \leq i \leq n-1, i < j.$$

dissipation and friction :

Rayleigh's dissipation function known as

$$D = \frac{1}{2}d_1\dot{\theta}_1^2 + \sum_{k=2}^n \frac{1}{2}d_k (\dot{\theta}_k - \dot{\theta}_{k-1})^2 + \frac{1}{2}d_0\dot{x}^2 \quad (1.15)$$

where d_i , for $i = 0, \dots, n$ denotes the friction coefficient in the i -th joint.

Lagrange equation We define $q_i = \theta_i, i = 0, \dots, n, q_{n+1} = x, \dot{q}_0 = 0$ and w denotes the external force on the cart, then for $L = K - P$ so

$$\begin{aligned} 0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = k_i \ddot{q}_i + \sum_{j=1, j \neq i}^n \mu_{ij} \ddot{q}_j \cos(q_i - q_j) \\ &+ \sum_{j=1, j \neq i}^n \mu_{ij} \dot{q}_j^2 \sin(q_i - q_j) + v_i \cos q_i \ddot{q}_{n+1} - v_i g \sin q_1 \\ &+ d_i(\dot{q}_i - \dot{q}_{i-1}) - d_{i+1}(\dot{q}_{i+1} - \dot{q}_i) \end{aligned} \quad (1.16)$$

$$\begin{aligned} f &= \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{n+1}} - \frac{\partial L}{\partial q_{n+1}} + \frac{\partial D}{\partial \dot{q}_{n+1}} \\ &= \sum_{i=1}^n v_i \ddot{q}_i \cos q_i - \sum_{i=1}^n v_i g \sin q_i + \ddot{q}_{n+1} \left(\sum_{i=0}^n m_i \right) + d_0 \ddot{q}_{n+1} \end{aligned}$$

Because, concerning x , equation (1.16) depends only on its second derivative \ddot{x} , we can consider that the cart acceleration represents the input u of the n link inverted pendulum, i.e. $\ddot{x} = u$. The equation for x can be completely omitted in this case. By reordering the terms in (1.16) we obtain the matrix model

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q)q = V(q)u \quad (1.17)$$

where q is the n -th vector of the joint angles and the matrices, $M(q), C(q, \dot{q}), G(q), V(q)$

are given by

$$M(q) = \begin{bmatrix} M_{11} & M_{12} & \cdots & \cdots & M_{1n} & 0 \\ M_{21} & M_{22} & \cdots & \cdots & M_{2n} & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & M_{n-1n-1} & M_{n-1n} & 0 \\ M_{n1} & \cdots & \cdots & M_{nn-1} & M_{nn} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

with

$$(M_{11} = k_1); (M_{12} = \mu_{12} \cos(q_1 - q_2)); (M_{1n} = \mu_{1n} \cos(q_1 - q_n))$$

$$(M_{21} = \mu_{12} \cos(q_1 - q_2)); (M_{22} = k_2); (M_{2n} = \mu_{2n} \cos(q_2 - q_n))$$

$$(M_{n1} = \mu_{1n} \cos(q_1 - q_n)); (M_{nn-1} = \mu_{n-1n} \cos(q_{n-1} - q_n)); (M_{nn} = k_n)$$

$$(M_{n-1n} = \mu_{n-1n} \cos(q_{n-1} - q_n)); (M_{n-1n-1} = k_{n-1})$$

$$C(q, \dot{q}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} & 0 \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2n} & \vdots \\ C_{31} & C_{32} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & C_{n-1n-1} & C_{n-1n} & 0 \\ C_{n1} & \cdots & \cdots & C_{nn-1} & C_{nn} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

with

$$(C_{11} = d_1 + d_2); (C_{12} = -d_2 + \mu_{12}\dot{q}_2 \sin(q_1 - q_2)); (C_{13} = \mu_{13}\dot{q}_3 \sin(q_1 - q_3))$$

$$(C_{1n} = \mu_{1n}\dot{q}_n \sin(q_1 - q_n)); (C_{21} = -d_2 - \mu_{12}\dot{q}_1 \sin(q_1 - q_2))$$

$$(C_{22} = d_2 + d_3); (C_{23} = -d_3 + \mu_{23}\dot{q}_3 \sin(q_2 - q_3)); (C_{2n} = \mu_{2n}\dot{q}_n \sin(q_2 - q_n))$$

$$(C_{31} = -\mu_{13}\dot{q}_1 \sin(q_1 - q_3)); (C_{32} = -d_3 - \mu_{13}\dot{q}_1 \sin(q_1 - q_3))$$

$$(C_{n-1n-1} = d_{n-1} + b_n); (C_{n-1n} = -b_n + \mu_{n-1n}\dot{q}_n \sin(q_{n-1} - q_n))$$

$$(C_{n1} = -\mu_{1n}\dot{q}_1 \sin(q_1 - q_n)); (C_{nn-1} = -b_n - \mu_{n-1n}\dot{q}_{n-1} \sin(q_{n-1} - q_n); (C_{nn} = d_n)$$

$$G(q) = \text{diag} \left(\frac{-v_1 g \sin q_1}{q_1}, \frac{-v_2 g \sin q_2}{q_2}, \dots, \frac{-v_n g \sin q_n}{q_n}, 0 \right)$$

$$V = \begin{bmatrix} -v_1 \cos q_1 & \cdots & -v_n \cos q_n & 1 \end{bmatrix}^T$$

The dynamic model (1.17) is linear in the r-vector

$$\begin{aligned} \tilde{\delta} = & [\mu_{12}, \mu_{13}, \dots, \mu_{1n}, \mu_{23}, \dots, \mu_{2n}, \dots, \mu_{n-1n}, \\ & , k_1, \dots, k_n, v_1, \dots, v_n, d_1, \dots, d_n]^T \end{aligned} \quad (1.18)$$

containing the parameters introduced in Propositions 1, 2, and in equation (1.15). Thus, (1.17) can be expressed in the form

$$\tilde{\Phi}(q, \dot{q}, \ddot{q}, u) \tilde{\delta} = 0 \quad (1.19)$$

where $\tilde{\Phi}(q, \dot{q}, \ddot{q}, u) = [\tilde{\Phi}_1 \tilde{\Phi}_2 \tilde{\Phi}_3 \tilde{\Phi}_4]$ is $n \times r$ matrix, depending on the joint angles q , velocities \dot{q} , accelerations \ddot{q} and the acceleration u of the cart with

$$\tilde{\Phi}_1 = \begin{bmatrix} f_{1,2} & f_{1,3} & \cdots & f_{1,n} & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ f_{2,1} & 0 & \cdots & 0 & f_{2,3} & f_{2,4} & \cdots & f_{2,n} & \cdots & 0 \\ 0 & f_{3,1} & & & f_{3,2} & 0 & \cdots & 0 & \cdots & \vdots \\ \vdots & & \ddots & & & \ddots & & & \cdots & f_{n-1,n} \\ 0 & & & f_{n,1} & 0 & & f_{n,2} & \cdots & f_{n,n-1} \end{bmatrix}$$

where

$$f_{i,j} = f(q_i, q_j) = \cos(q_i - q_j)\ddot{q}_j + \sin(q_i - q_j)\dot{q}_j^2, \text{ for } i, j = 1, \dots, n, i \neq j,$$

$$\tilde{\Phi}_2 = \text{diag}(g_1, \dots, g_n), \tilde{\Phi}_3 = \text{diag}(h_1, \dots, h_n),$$

where

$$g_i = \ddot{q}_i, h_i = u \cos q_i - g \sin q_i, \text{ with } i = 1, \dots, n,$$

$$\tilde{\Phi}_4 = \begin{bmatrix} k_1 & -k_{1,2} & 0 & \cdots & 0 \\ 0 & k_{1,2} & -k_{2,3} & \cdots & 0 \\ \vdots & 0 & k_{2,3} & \ddots & \\ \vdots & & \ddots & \ddots & -k_{n-1,n} \\ 0 & & & 0 & k_{n-1,n} \end{bmatrix}$$

$$k_1 = \dot{q}_1 \text{ and } k_{i,j} = \dot{q}_j - \dot{q}_i, \text{ for } i = 1, \dots, n-1, j = i+1.$$

We obtain a new system of equations if we divide each equation of the system 1.19 by μ_{n-1n} . the new system form is

$$\Phi(q, \dot{q}, \ddot{q}, u)\delta = \Psi(q, \dot{q}, \ddot{q}, u) \tag{1.20}$$

The equation (1.20) represents the regression model, where $\Phi(q, \dot{q}, \ddot{q}, u)$ is $n \times (r-1)$ matrix, δ is $(r-1)$ vector of parameters, and $\tilde{\Phi}(q, \dot{q}, \ddot{q}, u)$ is n vector, and all these symbols are given by (15), the dimension $(r-1)$ of the vector δ is equal to $3n + \frac{n(n-1)}{2} - 1$.

$$\Phi(q, \dot{q}, \ddot{q}, u) = \begin{bmatrix} f_m & g_m & h_m & k_m \end{bmatrix}$$

with

$$f_m = \begin{bmatrix} f_{1,2} & f_{1,3} & \cdots & \cdots & f_{1,n} & 0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 & 0 \\ f_{2,1} & 0 & \cdots & \cdots & 0 & f_{2,3} & f_{2,4} & \cdots & \cdots & f_{2,n} & \cdots & 0 & 0 \\ 0 & f_{3,1} & & & f_{3,2} & 0 & \cdots & & & 0 & \cdots & \vdots & \vdots \\ \vdots & & \ddots & & & \ddots & & & & \cdots & f_{n-2,n-1} & f_{n-2,n} \\ 0 & & & f_{n-1,1} & 0 & & & f_{n-1,2} & \cdots & f_{n-1,n-2} & 0 \\ 0 & & & & f_{n,1} & 0 & & & f_{n,2} & \cdots & 0 & f_{n,n-2} \end{bmatrix}$$

$$g_m = \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & & \\ \vdots & & \ddots & \\ \vdots & & & g_n \\ 0 & & & & \end{bmatrix}, h_m = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_2 & & \\ \vdots & & \ddots & \\ \vdots & & & h_n \\ 0 & & & & \end{bmatrix}$$

$$k_m = \begin{bmatrix} k_1 & -k_{12} & 0 & \cdots & 0 \\ 0 & k_{12} & -k_{23} & & \\ \vdots & & \ddots & \ddots & \\ & & & & -k_{n-1n} \\ 0 & & & & k_{n-1n} \end{bmatrix}$$

$$\delta = \frac{1}{\mu_{n-1n}} \begin{bmatrix} \mu_{12} & \mu_{13} & \cdots & \mu_{1n} & \mu_{23} & \cdots & \mu_{2n} & \cdots & \mu_{n-2,n-1} & \mu_{n-2,n} \\ k_1 & \cdots & k_n & v_1 & \cdots & v_n & d_1 & \cdots & d_n \end{bmatrix}^T$$

$$\Psi(q, \dot{q}, \ddot{q}, u) = \begin{bmatrix} 0 & 0 & \cdots & -f_{n-1,n} & -f_{n,n-1} \end{bmatrix}^T$$

1.5 Conclusion

The equation of motion is a mathematical equation describing the motion of a physical object. In general, the equation of motion includes the acceleration of the object as a function of its position, velocity, mass, and any variables affecting any of these. And the state representation makes it possible to model a dynamic system using state variables. This representation, which can be linear or not, continuous or discrete, makes it possible to determine the state of the system at any future instant if one knows the state at the initial instant and the behavior of the exogenous variables which influence the system.

Chapter 2

Inverted pendulum control

2.1 Introduction

A swing-up controller is used to overcome the non-linearity of the system. Since the system is linearised near the equilibrium position, swing-up tries to bring the system near the equilibrium position. Initially, the cart pendulum system is presented in a state space form using generalized coordinates. Afterward, the total energy of the system is defined. It is used in the formulation of the control law. Eventually, using LaSalle's invariance principle the control law is obtained.

2.2 Controllability

we want to control our system, we need to construct laws of control and that is not possible only if our system is controllable, if not, the non-controllable modes have to be stable.

2.2.1 Controllability of the non-linear system

the inverted pendulum on a cart is a non-linear system in the form of

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x)$$

we extract the controllability matrix as

$$C_{fg} = \begin{bmatrix} g(x) & ad_f g(x) & ad_f^2 g(x) & \cdots & ad_f^{n-1} g(x) \end{bmatrix}$$

where $ad_f g(x)$ is the hook lie of f and g defined as

$$\begin{aligned} ad_f g(x) &= [f, g] = L_f g - L_g f \\ &= \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \end{aligned}$$

where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are the jacobiennes matrix So to say the system non-linear is controllable it must the $\det(C_{fg}) \neq 0$ which means the rank of C_{fg} equal to n

2.2.2 Controllability of the linear systems

after defining the equilibrium we can make a linear approximation around equilibrium points so our system becomes this form

$$\dot{x} = Ax + Bu$$

$$y = cx$$

we can control it but it is necessary to be controllable, the controllability matrix of kalman is

$$C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

to say it's controllable the rank of C must be n.

2.3 Observability

The observability of the system is important in the full-state estimation

2.3.1 Observability of the linear systems

The observability matrix of kalman in the linear systems is

$$O = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}$$

to say the system is observable the observability matrix must be of rank n .

2.4 One inverted pendulum swing up control

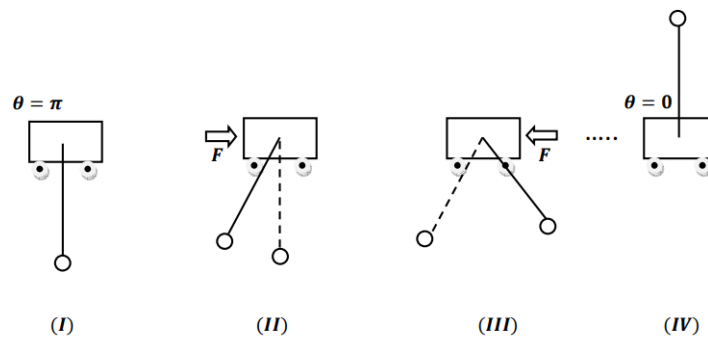


Figure 2.1: Swing up

The swing up process is described in fig 2.1 in (I) the pendulum is in the stable equilibrium point, in (II) after applying a force (F) on the cart you see the pendulum swing a little bit, after we apply a force in the other direction so the pendulum swings more and more until it reaches the stabilisation range and with the help of the control law we stabilise it [9]. We use a strategy that controls the total amount of energy in the system to swing up the pendulum to its upright position [6], The total energy of the system $E(q, \dot{q})$ is defined as the sum of its kinetic and potential energies, $(K(q, \dot{q}))$ and $(P(q))$. In the origin without the control ($u = 0$), $P(q)$ is chosen to be

zero $P(0) = 0$ but, because of conditions imposed on the derivative of the Lyapunov function, as described later, an offset $P_{off} = 3m_1gl$ has been added. The offset P_{off} has been chosen big enough (but cautiously) to ensure positive total energy and not introduce substantial modifications in the system. So the energy of the system in the initial is $P(0) + P_{off}$. The total energy of the inverted pendulum considering (3.23) is:

$$\begin{aligned}
 E(q, \dot{q}) &= K(q, \dot{q}) + P(q) + P_{off} \\
 &= \frac{1}{2} \begin{bmatrix} \dot{x} & \dot{\theta}_1 \end{bmatrix} M(\theta) \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \end{bmatrix} + m_1gl(\cos \theta_1 - 1) + 3m_1gl
 \end{aligned} \tag{2.1}$$

Equilibrium points :

a point $x = x^*$ in the state, space is said to be an equilibrium point (or singular point, fixed point, stationary state) of the system if the state initiated at x^* will stay in x^* for all future times. We can get the equilibrium points by $\dot{x} = 0$ so

$$\dot{x}_1 = \dot{x} = 0$$

$$\dot{x}_2 = 0 \Rightarrow \sin \theta_1 = 0 \Rightarrow \theta_1 = i\pi$$

$$\dot{x}_3 = 0 \Rightarrow \dot{\theta}_1 = 0$$

$$\dot{x}_4 = 0 \Rightarrow \ddot{\theta}_1 = 0$$

So there are two equilibrium points

$$x_1^* = (\alpha_x, 0, 0, 0)$$

$$x_2^* = (\alpha_x, 0, \pi, 0)$$

where (α_x) can be any possible value of the cart displacement x . or we can get them by making the derivative of $P(q)$ equal to zero

$$\frac{\partial P(q)}{\partial q} = 0 \Rightarrow -m_1 g l \sin \theta_1 = 0.$$

Linearisation around the equilibrium points :

The linearisation around the equilibrium points helps us to make study the local behaviour of the system. The linearisation is obtained by using the Taylor expansion and neglecting the higher order terms for small variations around the equilibrium point. We need to find the Jacobian matrix of f , let A be the Jacobian matrix evaluated at the points of equilibrium, So

$$J_f = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{x=x^*} = \left. \frac{\partial f}{\partial x} \right|_{x=x^*}$$

$$B = \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_3} \\ \frac{\partial g}{\partial x_4} \end{bmatrix}_{x=x_1^*}$$

$$B_1 = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{lM} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{lM} \end{bmatrix}$$

with $a_{ij} = \frac{\partial f_i}{\partial x_j}$ $i \in N$ we find

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix}$$

where

$$\frac{\partial f_2}{\partial x_2} = \frac{-d_0(M + m_1 \sin^2 x_3)}{(M + m_1 \sin^2 x_3)^2}$$

$$\frac{\partial f_2}{\partial x_3} = \frac{(M + m_1 \sin^2 x_3) [m_1 \cos x_3 (lx_4^2 - g \cos x_3) + m_1 g \sin^2 x_3]}{(M + m_1 \sin^2 x_3)^2}$$

$$\frac{-(m_1 \sin 2x_3) [m_1 \sin x_3 (lx_4^2 - g \cos x_3)]}{(M + m_1 \sin^2 x_3)^2}$$

$$\frac{\partial f_2}{\partial x_4} = \frac{(M + m_1 \sin^2 x_3)(2lx_4 m_1 \sin x_3)}{(M + m_1 \sin^2 x_3)^2}$$

$$\frac{\partial f_4}{\partial x_2} = \frac{m_1 l^2 (M + \sin^2 x_3)(m_1 l d_0 \cos x_3)}{(m_1 l^2 (M + \sin^2 x_3))^2}$$

$$\frac{\partial f_4}{\partial x_3} = \frac{m_1 l^2 (M + \sin^2 x_3) [-m_1^2 l^2 x_4^2 (\cos^2 x_3 - \sin^2 x_3) - m_1 l d_0 x_2 \sin x_3 + (M + m_1) g m_1 l \cos x_3]}{(m_1 l^2 (M + \sin^2 x_3))^2}$$

$$\frac{(m_1 l^2 \sin 2x_3)(-m_1^2 l^2 \sin x_3 \cos x_3 x_4^2 + m_1 l d_0 \cos x_3 x_2 + (M + m_1) g m_1 l \sin x_3)}{(m_1 l^2 (M + \sin^2 x_3))^2}$$

$$\frac{\partial f_4}{\partial x_4} = \frac{((m_1 l^2 (M + \sin^2 x_3))(-2m_1^2 l^2 \sin x_3 \cos x_3 x_4))}{(m_1 l^2 (M + \sin^2 x_3))^2}$$

$$A_1 |_{x=x_1^*} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-d_0}{M} & \frac{-m_1 g}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{d_0}{lM} & \frac{(M+m_1)g}{M} & 0 \end{bmatrix}$$

$$A_2 |_{x=x_2^*} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-d_0}{M} & \frac{-m_1 g}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-d_0}{lM} & \frac{-(M+m_1)g}{M} & 0 \end{bmatrix}$$

Those equilibria result to be $[q, \dot{q}]^T = [i\pi, 0], i \in Z$. The second derivative of $P(q)$ instead is negative for $q = 0$ and positive for $q = \pi$. Origin corresponds then to a local maximum of the potential energy (unstable) while $q = \pi$ is a minimum (stable).

We wrote that the potential energy in the upright position is $P(0) + P_{off} = 0 + 3m_1gl$ while in the downright position, the distance become $2l$ from the reference, so the potential energy equal to $3m_1gl - 2m_1gl = m_1gl$. So, it is $P(q) \in [m_1gl, 3m_1gl]$.

As respects initial kinetic energy, it is proportional to the squared value of the velocity and it assumes its minimum value when velocity is zero. The potential the energy of the inverted pendulum is defined as

$$P(q) = m_1gl(\cos \theta_1 - 1)$$

and its derivative is related to the $G(q)$ as

$$G(q) = \frac{\partial P}{\partial q} \tag{2.2}$$

From equations (1.1)-(1.5) and (1.7)-(2.2) it is obtained the rate of the energy change of system

$$\begin{aligned}
 \dot{E} &= \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T G(q) \\
 &= \dot{q}^T \left(-C(q, \dot{q})\dot{q} - D\dot{q} - G(q) + F + \frac{1}{2}\dot{M}(q)\dot{q} \right) + \dot{q}^T G(q) \\
 &= \dot{q}^T \left(-C(q, \dot{q})\dot{q} - D\dot{q} + \frac{1}{2}\dot{M}(q)\dot{q} \right) + \dot{q}^T F \\
 &= \frac{1}{2}\dot{q}^T (\dot{M}(q)\dot{q} - 2c(q, \dot{q}))\dot{q} + \dot{q}^T F - \dot{q}^T D\dot{q} \\
 &= \dot{q}^T F - \dot{q}^T D\dot{q} \\
 &= \dot{q}^T \begin{bmatrix} u \\ 0 \end{bmatrix} - \dot{q}^T \begin{bmatrix} d_0 & 0 \\ 0 & d_1 \end{bmatrix} \dot{q} \\
 &= \dot{x}u - \dot{x}^2 d_0 - \dot{\theta}_1^2 d_1
 \end{aligned} \tag{2.3}$$

The behavior of our system can be represented in the phase plane, where variables are angular position (θ) and angular velocity ($\dot{\theta}$).

NB: The phase plane method refers to graphically determining the existence of limit cycles in the solutions of the differential equation.

At any instant of time, the system is characterized by a certain pair (q, \dot{q}) due to its motion. This variable pair traces out a phase plane orbit which corresponds to a specific total energy E according to (2.5). Different initial conditions for the system give different values of total energy E , hence different orbits. The behavior of the system can be shown through its phase portrait which can be used to obtain the various values of the total energy E of the system. Due to the potential energy offset (P_{off}), the total energy will now converge to $3m_1gl$ when q and \dot{q} converge to zero. Substituting matrices from (1.11) in equation (2.5) and considering $\dot{x} = 0$ and $E(q, \dot{q}) = 3m_1gl$, for the case of an upright stabilized inverted pendulum it is obtained

$$\underbrace{K(q, \dot{q}) + P(q)}_{=0} + P_{off} = \frac{1}{2}\dot{\theta}_1^2 m_1 l^2 + m_1 gl (\cos \theta_1 - 1) + P_{off}$$

$$\frac{1}{2}\dot{\theta}_1^2 m_1 l^2 = -m_1 g l (\cos \theta - 1) \quad (2.4)$$

which expresses the homoclinic orbit

NB: a homoclinic orbit is a trajectory of a flow of a dynamical system that joins a saddle equilibrium point to itself.

Also, when $\theta_1 = 0$ (then $\dot{\theta}_1 = 0$), the the pendulum has reached an equilibrium point. Generally, when the pendulum is in the region $\theta_1 \in [0, 2\pi]$, the system (1.1) has two sets of equilibrium points: $(x, \dot{x}, \theta_1, \dot{\theta}_1) = (\alpha_x, 0, 0, 0)$ for the unstable equilibrium points, $(x, \dot{x}, \theta_1, \dot{\theta}_1) = (\alpha_x, 0, \pi, 0)$ for the stable equilibrium points, If the system can be brought to the homoclinic orbit then the task of swinging up the pendulum has been solved since the pendulum, eventually, will approach an unstable equilibrium point if it follows the homoclinic orbit. Finally, the swing up control will be switched to a linear controller which will ensure (local) asymptotic stability for this equilibrium point and will stabilize the pendulum in the upright vertical position

2.5 One inverted pendulum stabilization control

Lyapunov stability Our system is

$$\dot{x} = f(x) \quad (2.5)$$

where $x \in \mathfrak{R}$ and for which we assume that $f(0) = 0$, i.e. the origin is an equilibrium of the system. One of the most important properties related to this equilibrium is its stability, and assessing whether the stability of the equilibrium holds has been at the center of systems and control research for more than a century. Lyapunov formulated sufficient conditions for the stability that does not require knowledge of the solution, but is based on the construction of a function, well known nowadays as a "Lyapunov function". More precisely, the conditions are : For an open set $D \subset \mathfrak{R}^n$ with $0 \in D$, suppose there exists a continuously differentiable function $V : D \rightarrow \mathfrak{R}$ such that

$$V(0) = 0$$

$$V(x) > 0$$

$\forall x \in D \setminus \{0\}$

$$\frac{\partial V(x)}{\partial x} f(x) \leq 0$$

$\forall x \in D$ Then $x = 0$ is stable, moreover, if

$$\frac{\partial V(x)}{\partial x} f(x) < 0$$

$\forall x \in D$ then $x=0$ is asymptotically stable.

The proposed candidate function is expressed as the sum of squares of total energy, position and velocity of the cart. The purpose is to have all the terms of $V(q, \dot{q})$ equal to zero in the upright position:

$$V = \frac{K_E}{2}(E - p_{off})^2 + \frac{K_v}{2}\dot{x}^2 + \frac{K_x}{2}x^2 \quad (2.6)$$

where gains K_E, K_v, K_x are strictly positive constants. Lyapunov's stability theorem requires the candidate function (2.6) to be positive definite but from the expression of energy (2.5) it is possible to get $E - P_{off} = 0$ by the combination of θ and $\dot{\theta}$ values other than zero. Therefore, (2.6) is not a Lyapunov function. Failing to design a Lyapunov function does not preclude alternative approaches: conditions in Lyapunov's stability theorem are only sufficient

Lyapunov-LaSalle Let $V : D \rightarrow \mathfrak{R}^+$ a function admitting continuous partial derivatives such that there exists l for which the region D_l defined by $V(x) < l$ is bounded and $\dot{V}(x) \leq 0$ for all $x \in D_l$. Let $R = \{x \in D_l : \dot{V}(x) = 0\}$ and let M be the largest positively invariant set included in R . Then any solution from D_l tends to M when $t \rightarrow \infty$. In particular if 0 is the only orbit contained in R then $x = 0$ is asymptotically stable and D_l is contained in its basin of attraction. The set D_l represents an estimation for the region of attraction to the equilibrium set

LaSalle's theorem allows us to use function (2.6) also if it is not positive definite, there are conditions only on the derivative of $V(q, \dot{q})$. It extends Lyapunov's theorem and can be used when instead of an isolated equilibrium point, the system has an equilibrium set. Using LaSalle's theorem principle, a control law will be designed to bring the pendulum to the invariant set M , starting from a region of attraction

2.5.1 Control Law

The terms $d_0\dot{x}$ will disappear, because the friction will be compensated by u so the equation 2.3 and as we supposed in chapter 1 the friction at the joint equal to zero becomes [6]

$$\dot{E} = \dot{x}u \quad (2.7)$$

the derivative of lyapunov's function becomes

$$\begin{aligned} \dot{V} &= K_E(E - P_{off})\dot{E} + K_v\dot{x}\ddot{x} + K_x x\dot{x} \\ &= K_E(E - P_{off})\dot{x}u + K_v\dot{x}\ddot{x} + K_x x\dot{x} \\ &= \dot{x} [K_e(E - P_{off})u + K_v\ddot{x} + K_x x] \end{aligned} \quad (2.8)$$

To simplify our system let make

$$\frac{m \sin \theta_1 (l\dot{\theta}_1^2 - g \cos \theta_1) + u}{M + m \sin^2 \theta_1} = \frac{\alpha(\theta_1, \dot{\theta}_1) + u}{\beta(\theta)_1} \quad (2.9)$$

By substituting equation 2.9 into 2.8 we get

$$\dot{V} = \dot{x} \left[\underbrace{K_E(E - P_{off})u + K_v \left(\frac{\alpha(\theta_1, \dot{\theta}_1) + u}{\beta(\theta)_1} \right) + K_x x}_{-K_\delta \dot{x}} \right] \quad (2.10)$$

$$-K_\delta \dot{x} = u \left(K_E(E - P_{off}) + \frac{K_v}{\beta(\theta)_1} \right) + K_v \frac{\alpha(\theta, \dot{\theta})}{\beta(\theta)_1} + K_x x \quad (2.11)$$

We can deduce u

$$u = \frac{\beta(\theta_1)(-K_\delta \dot{x} - K_x x) - K_v \alpha(\theta, \dot{\theta})}{\beta(\theta_1)K_E(E - P_{off}) + K_v} \quad (2.12)$$

The control u is defined if we avoid singularities in 2.11

$$K_E(E - P_{off}) + \frac{K_v}{\beta(\theta_1)} \neq 0$$

$$\frac{K_v}{K_E} \neq -(E - P_{off})(M + m \sin^2 \theta_1) \quad (2.13)$$

we suppose the term $(E - P_{off})$ is not smaller than $-2m_1gl$

$$E - P_{off} \geq -2m_1gl \quad (2.14)$$

From 2.13 and 2.14, the following constrain is achieved

$$\begin{aligned} \frac{K_v}{K_E} &> 2m_1gl \max_{\theta} \{M + m_1 \sin^2 \theta\} \\ \Rightarrow \frac{K_v}{K_E} &> 2m_1gl(M + m_1) \end{aligned} \quad (2.15)$$

The ratio of K_v to K_E will be provided according to inequality 2.15. Substituting 2.11 in 2.10 it is obtained

$$\dot{V} = -K_\delta \dot{x}^2 - \theta_1^2 d_1 K_E (E - P_{off}) \quad (2.16)$$

with $d_1 = 0$ the equation become

$$\dot{V} = -K_\delta \dot{x}^2$$

where $K_\delta > 0$ to ensure that $\dot{V}(q, \dot{q})$ is negative semi definite as required in the LaSalle's invariance principle.

2.6 Control strategies

Away from the control-based energy, there are many controllers we can apply to our approximated system around the equilibrium points as

2.6.1 Pole placement control

The pole placement method is undoubtedly a very powerful design tool, especially for the preliminary assessment of the feedback gains needed to achieve a specified level of closed-loop stability. our approximated system is

$$\dot{x} = Ax + Bu$$

where the pair (A,B) is stabilize, are often stabilized by designing a linear feedback control law $u = -Kx$ such that the matrix K is chosen so that the eigenvalues of the closed-loop matrix $[A - BK]$ are placed arbitrarily in the left-half plane.

$$Re[eig[A - Bk]] < 0$$

so, the feedback gain $K = [k_1, k_2]$ can be designed conveniently in order to place the poles anywhere in the left-half plane, hence the control $u(t)$ will drive the system response from a set of initial conditions $x(0)$ to zero as $t \rightarrow 0$. NB: we get out system eigenvalues by solving

$$\det(A - \lambda I) = 0$$

2.6.2 Linear Quadratic regulation

The linear quadratic regulator is likely the most important and influential result in optimal control theory to date. It is considered as a feedback controller. Nevertheless, LQR computes the control signals by modelling it as an optimization problem. That is, it tries to compute control

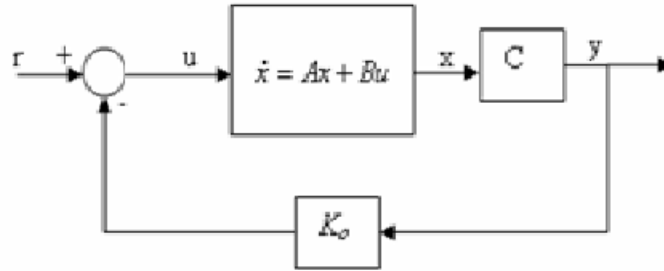


Figure 2.2: Stabilisation with pole placement feedback control

signals that minimizes some cost function. We have our approximated system

$$\dot{x} = Ax + Bu$$

we have the desired state we want to reach x_d . Our job is to find a sequence of control signals u that will take us to this goal state. The assumption we make in LQR is that the controller is a linear controller. It turns out that in most cases, a simple linear controller as in the pole placement in the form of $u = -kx$ to define k there are some steps. First, we need to define a cost function that quantifies the performance of the controller. In LQR we define the cost function as a quadratic cost function:

$$J = \int_0^{\infty} [(x_t - x_d)^T Q (x_t - x_d) + u_t^T R u_t] dt$$

while Q and R are positive definite matrices where off-diagonal elements in the matrix are all 0. Each of the diagonal elements in Q reflects how much we care about a particular error signal and R reflects how much we care about performing a particular action. We choose them manually. Now that we have defined the cost function, our job is to find the best sequence of u that minimizes this cost. Since $u = -kx$ finding k is the same as finding u so one of the common methods to solve for k is by first solving Riccati equation

$$\dot{y} = a(x)y + b(x)y^2 + c(x)$$

where $a(x), b(x), c(x)$ are continuous functions of x .

2.6.3 Kalman filter

Kalman filtering is an algorithm that provides estimates of some unknown variables given the measurements observed over time [10]. Kalman filters are used to estimate states based on linear dynamical systems in state -space format.

$$x_k = Fx_{k-1} + Buk - 1 + W_{k-1}$$

where F is the state transition matrix applied to the previous state vector x_{k-1} , B is the control-input matrix applied to the control vector u_{k-1} , and W_{k-1} is the process noise vector (disturbances) The measurement model is known as:

$$z_k = Hx_k + v_k$$

where z_k is the measurement vector, H is the measurement matrix, v_k is the measurement noise vector. consequently, the kalman filter provides an estimate of c_k at time k , you enter the measures $z_1, z_2 \dots z_k$ and the information of the system described by F, B, H, Q and R .

The applications of kalman filter

you can find the kalman filter in many process such as [11]

- Tracking objects
- Economics
- Navigation
- Image processing
- State and parameter estimation (as in the inverted pendulum system)
- Data merging and others [12]

2.7 Double inverted pendulum control

Matrix form : if we suppose that the friction and the disturbance can be neglected, then we can write the dynamic that we get in chapter 1 as this matrix form [8]

$$D(\theta)\ddot{y} + C(\theta, \dot{\theta})\dot{y} + G(\theta) = Hu$$

where $D(\theta)$ is symmetric and non-singular and equal to

$$D(\theta) = \begin{bmatrix} M + m_1 + m_2 & l_1(m_1 + m_2) \cos \theta_1 & m_2 l_2 \cos \theta_2 \\ l_1(m_1 + m_2) \cos \theta_1 & l_1^2(m_1 + m_2) + I_1 & m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \\ m_2 l_2 \cos \theta_2 & m_2 l_1 l_2 \cos(\theta_1 - \theta_2) & m_2 l_2^2 + I_2 \end{bmatrix}$$

and

$$C(\theta, \dot{\theta}) = \begin{bmatrix} 0 & -l_1(m_1 + m_2) \sin \theta_1 \dot{\theta}_1 & -m_2 l_2 \sin \theta_2 \dot{\theta}_2 \\ 0 & 0 & m_2 l_1 l_2 \sin(\theta_1 - \theta_2) \dot{\theta}_2 \\ 0 & -m_2 l_1 l_2 \sin(\theta_1 - \theta_2) \dot{\theta}_1 & 0 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} 0 \\ -(m_1 + m_2)gl_1 \sin \theta_1 \\ -m_2 gl_2 \sin \theta_2 \end{bmatrix}$$

Cost function we define the state vector $x \in \mathfrak{R}^6$ as

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

the dynamic equation can be given as

$$\dot{x} = \begin{bmatrix} 0 & I \\ 0 & -D^{-1}C \end{bmatrix} x + \begin{bmatrix} 0 \\ -D^{-1}G \end{bmatrix} + \begin{bmatrix} 0 \\ D^{-1}H \end{bmatrix} u \quad (2.17)$$

the design of an optimal control means the minimizing of an accumulative cost function J_t

where

$$J_t = \sum_{k=t}^{t_{final}} L_k(x_k, u_k) \quad (2.18)$$

the optimization is done with respect to the constraints of the system dynamics, in our system

$$L_k(x_k, u_k) = x_k^T Q x_k + u_k^T R u_k \quad (2.19)$$

Linear quadratic control to drive an approximate linear solution to the optimal control problem 2.18 2.19 we need to linearise the non-linear system, the linearisation around $x = 0$ gives

$$\dot{x} = Ax + Bu \quad (2.20)$$

where

$$A = \begin{bmatrix} 0 & I \\ -D(0)^{-1} \frac{\partial G(0)}{\partial \theta} & 0 \end{bmatrix} \quad (2.21)$$

$$B = \begin{bmatrix} 0 \\ D(0)^{-1} H \end{bmatrix} \quad (2.22)$$

thus the LQR solution is obtained by

$$u = -R^{-1} B^T P_c x = -K_c x \quad (2.23)$$

where P_c is a steady-state solution of the differential Riccati equation. To implement computerized digital control, dynamic equations (2.20) are approximately discretized as $\Phi \approx e^{A\Delta t}$, $\Gamma \approx B\Delta t$ and digital LQR control is then given

$$u_k = -R^{-1} \Gamma^T P x_k = -K x_k \quad (2.24)$$

where P is the steady state solution of the difference Riccati equation, obtained by solving

the discrete-time algebraic Riccati equation

$$\Phi^T [P - P\Gamma(R + \Gamma^T P\Gamma)^{-1}\Gamma^T P] \Phi - P + Q = 0 \quad (2.25)$$

where $Q \in \mathfrak{R}^{6 \times 6}$ and $R \in \mathfrak{R}$ are positive definite state and control cost matrices. Since linearisation (2.20)–(2.22) accurately represents the double inverted pendulum on a cart system (2.17) in the equilibrium, the LQR control (2.23) or (2.24) will be a locally near-optimal stabilizing control.

2.8 Conclusion

The purpose of the command would be to bring the pendulum back from its initial position to its position of unstable equilibrium and maintain it around this position. In general, the control problem consists of obtaining dynamic models of systems and using these models to determine control laws or strategies to achieve the desired system response and performance. Many control systems for stabilizing the inverted pendulum have been presented like the PID controller, pole placement feedback control, and linear quadratic regulator we talked about some of them in this chapter.

Chapter 3

Simulation

3.1 Introduction

The inverted pendulum system is one of the most popular experimental apparatuses used for control education. The simulation of the system has an important role to show us the reaction of the system and the best strategies to know which parameters and control laws are the best. By using the control law shown in the previous chapter we will try to simulate our system in MATLAB so we can see the system without a controller and after with one.

3.2 Open loop simulation

ODE45 solver is a standard solver for ordinary differential equations in matlab, it implements a Runge-Kutta method with a variable time step for efficient compilation. This solver is designed to handle the general problem

$$\frac{dx}{dt} = f(t, x)$$

with $x(t_0) = x_0$ and t is the independent variable, x is a vector of dependent variables [13]

NB Runge-Kutta method is a method for solving the initial-value problems of differential equations. it can be used to construct high order accurate numerical method by functions' self without needing the high order derivatives of functions [14].

To use the ODE45 function in MATLAB you need to divide your program into two scripts,

the first one is the function (you need to name the function as the name of the script) and in the second script you can call the function (the file of the function and the main script need to be in the same folder). We simulate the open loop (see fig 3.1) which mean our system without a controller it period of $t = 10s$ without friction and with $x_0 = \begin{bmatrix} 0 & 0 & 5\pi/6 & 0 \end{bmatrix}$

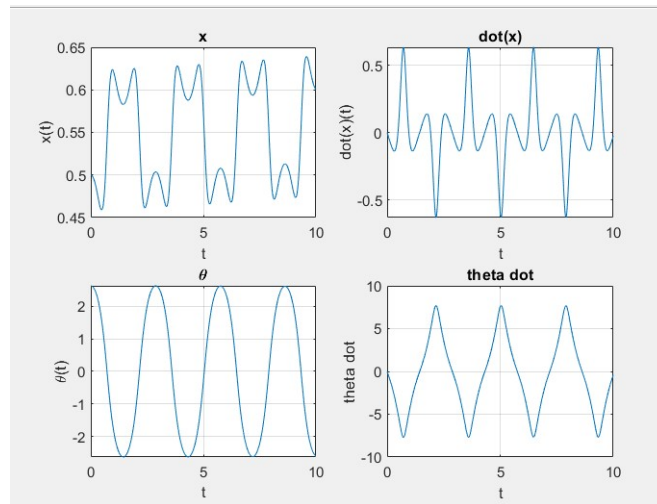


Figure 3.1: Simulation of an inverted pendulum on cart without controller

As we said our system is instable, so like you see in the fig 3.1 θ just oscillates, and the cart move with right and left.

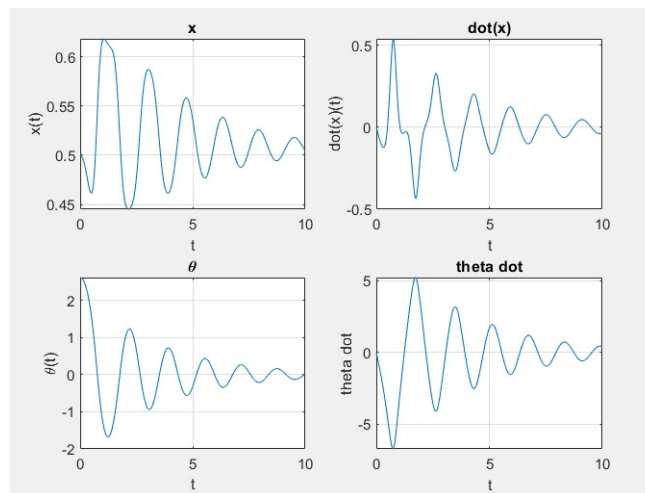


Figure 3.2: Simulation of the open loop with friction

After we did the same simulation with choosing the friction $d_0 = 1$ and $d_1 = 0.1$, you see (3.2) the deference in the angle θ . It starts oscillating at first but after the effect of the friction,

the oscillation starts damping until it reaches the stable equilibrium point.

Observation the pendulum is pointing downwards. In the absence of any control force, the system will naturally return to this state

3.2.1 Open loop poles

After the ode45 openloop test, you create a new script to write in it the approximated state space around the equilibrium points. We call them poles or eigenvalues of the matrix A , we can get them by applying a simple code

$$eig(A)$$

after applying this code we find those eigenvalues

$$\lambda_1 = 0$$

$$\lambda_2 = -0.0294$$

$$\lambda_3 = -3.9853$$

$$\lambda_4 = 3.9814$$

so we see that λ_2 and λ_3 are stable but λ_4 is unstable.

3.3 Stabilisation with Pole Placement

We talked about the pole placement control method in chapter 2 [15]. Now we will simulate it. In Matlab it is very easy to simulate this control method just by writing a simple code.

$$K = place(A, B, eigs)$$

where eigs are poles that had been chosen manually. Let chose the desired eigenvalue to be

$$\lambda_{d1} = -1$$

$$\lambda_{d2} = -1.5$$

$$\lambda_{d3} = -2.5$$

$$\lambda_{d4} = -2$$

3.3.1 Controllability test

Before we can apply the pole placement we need to see if our system is controllable or not, we talked about it theoretical but in Matlab we test it just by writing

$$ctrb(A, B)$$

then we see if the rank of our controllability function is under 4 degree, or we see if the $\det(ctrb(A, B)) \neq 0$ we found

$$rank(ctrb(A, B)) = 4$$

So our system (the approximation around equilibrium points) is controllable.

after placing our desired pole in Matlab we get our vector control K . where

$$K = \begin{bmatrix} 1.607 & 3.1250 & 2.83 & 11.8125 \end{bmatrix}$$

to check if our desired poles were placed we write this code

$$eig(A - BK)$$

we apply K in the ODE45 function we get our feedback control (see fig 3.3)

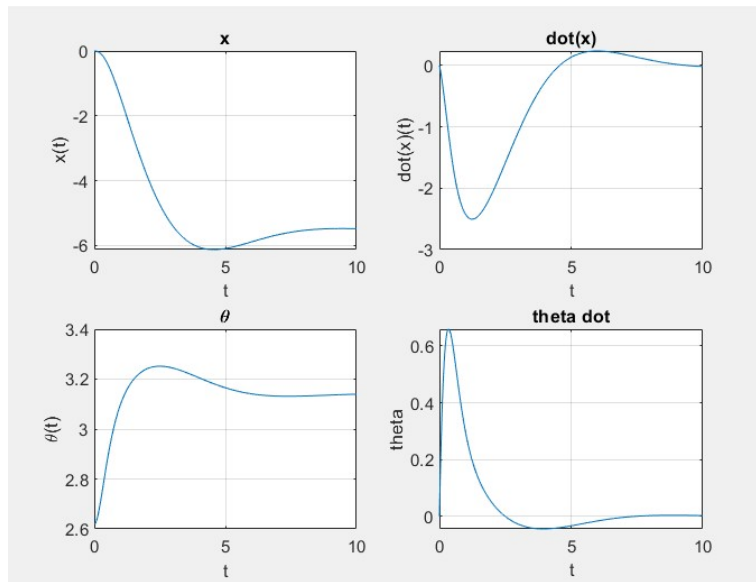


Figure 3.3: Simulation with pole placement feedback control

Here we don't concentrate on the position of the cart so as you see it's moving anywhere to stabilise the pendulum, in our example the cart stop at -5.5 . Observe in parallel at \dot{x} and θ you see at first we chose $\theta = 5\pi/6$ so the DC motor turn right in a fast way to pouch θ to become $\theta = 3.25$ Rad, and this angle is in the right of our desired angle so it turns left to stabilise in $\theta = \pi$

3.4 LQR simulation

In the pole placement method we choose the eigenvalues manually, but where are the best eigenvalues? you can spend a long time trying a lot of eigenvalues combinations, and how do we know if we found the best one. The linear quadratic regulator is amazing powerful tool in control theory [16], the idea is to build a cost function to tell me how bed is it if my state is really slow to converge to where i want it to. and also to build a cost function to the energy (maybe the input cost is much and expensive). in the second chapter, we explain that the cost function

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt$$

if my state is not where i tell it to be , Q tries to stabilize to zero so i make Q big. while we use DC motor so we don't care enough about input energy so we make R small. We define the matrix

Q and R in Matlab. In Matlab it is easy to make the LQR just by one line code

$$k = lqr(A, B, Q, R)$$

we get our control vector k

$$k_{LQR-1} = 31.6228$$

$$k_{LQR-2} = 54.9721$$

$$k_{LQR-3} = 281.6849$$

$$k_{LQR-4} = 69.29$$

after we see $eig(A - B * k)$ to see what LQR had chosen as eigenvalues we find

$$\lambda_{LQR-1} = -48.4475$$

$$\lambda_{LQR-2} = -1.0053$$

$$\lambda_{LQR-3} = -1.1 + 1.349i$$

$$\lambda_{LQR-4} = -1.1 + 1.349i$$

interestingly LQR takes one eigenvalue and makes it really aggressive λ_{LQR-1} . To see the most stabilising direction, we must our eigenvalue T and eigenvector D by using this code

$$[T, D] = eig(A - B * k)$$

our eigenvalues are the real part of D . the aggressive pole is the first one so we need to see the first column of T by using

$$T(:, 1)$$

$$T_{1,1} = 0.0119$$

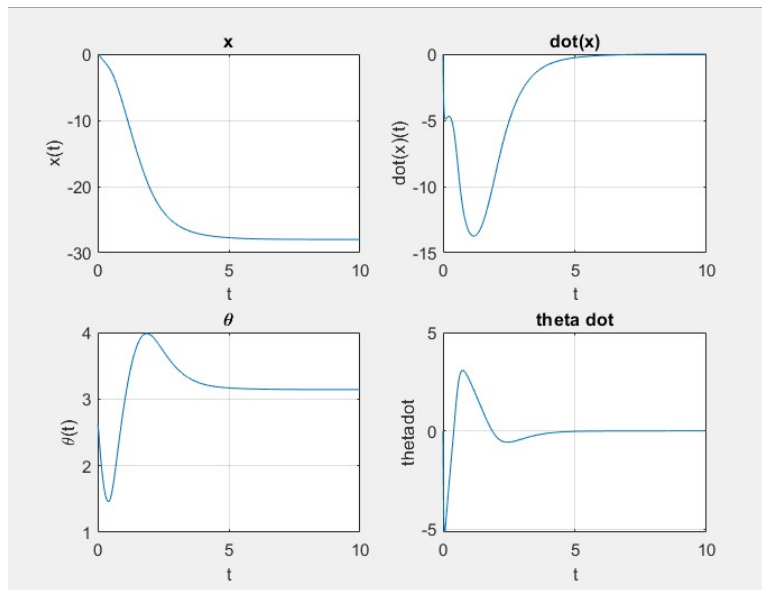


Figure 3.4: Simulation of the LQR responses

$$T_{2,1} = -0.5756$$

$$T_{3,1} = 0.0169$$

$$T_{4,1} = -0.8175$$

we observe that the most stabilising directions are the first and the third thus : are \dot{x} and $\dot{\theta}$ see fig 3.4

NB as you see , the regulator chooses the eigenvalues by itself i didn't tell him to do this, i just gave it a weight on how much i care about the stabilisation and energy and it tuned it for me.

Let's see how the system reacts when we enter the k_{LQR}

As we saw that the most stabilising directions are \dot{x} and $\dot{\theta}$ so in the fig 3.4 we see the response of \dot{x} does not have an overshoot as in the pole placement method, even the convergence time is shorter (in LQR took 4s and in Pole placement took 10s).

3.5 Kalman filter

we will develop Kalman filter to estimate the full state from a single measurement C [16], in our case we consider that we have only one encoder or a sensor to observe x (the position of the cart) so

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

In the up position, the system is unstable and here we are estimating to build the kalman filter, we are not doing a feedback control, so the instability of the pendulum in the upright position will pouch it to fall in the down, that means the only way to have kalman filter for the pendulum in the up position is to already be stabilising it in the upright position using full state feedback and that will happen when we combine the lqr and kalman filter in the next section but now we will study it when the pendulum in the down position.

3.5.1 observability test

As the controllability, there is simple code we write $obsv(A, C)$ to get our observability matrix and then we see $rank(obsv(A, C))$ we find it equal to 4 which mean it's observable.

3.5.2 Augment system

To build the kalman filter there are a few steps as we talked about in chapter 2 , we will build the augment system with disturbance and noise. We write in Matlab

$$V_d = 0.1 * eye(4)$$

$$V_n = 1$$

where V_d is the matrix of disturbance covariances, eye is identity matrix and V_n is the measurement noise covariances. as in the LQR V_d and V_n are the tuning knobs if i think that i have a big disturbance i make V_d bigger and if i think that i have big sensor noise i make V_n larger

now we build our augmented system B_F where

$$B_F = \begin{bmatrix} B & V_d & 0 * B \end{bmatrix}$$

next we will build the estimated state space system

$$sysC = ss(A, B_F, C, [00000 V_n])$$

then we will describe the system's full output by using

```
sysFullOutput = ss(A, B_F, eye(4), zeros(4, size(B_F, 2)));
```

we make $c = eye(4)$ because in the full output we measure all the system. we code the system's full output so we can simulate it and compare it to the system with noise.

3.5.3 Building Kalman filter

in Matlab it is easy to build the kalman filter, there is a command LQE which means linear quadratic estimator you give A , V_d C and V_n as inputs

$$[K_f, P, E] = lqe(A, V_d, C, V_d, V_n)$$

there is another method by choosing

$$K_f = (lqr(A^T, C^T, V_d, V_n))^T$$

here V_d and V_n can be as Q and R in LQR method. In our system to get the kalman gain matrix K_f by the second method finally, we define the state space for the kalman filter by

$$sysK_F = ss(\underbrace{A - K_f * C}_{A_k}, \underbrace{[BK_f]}_{B_k}, \underbrace{eye(4)}_{C_k}, \underbrace{0 * [BK_f]}_{D_k})$$

here $C_k = eye(4)$ because you measure all the states of kalman filter.

Simulate the estimated system :

we command the cart from (100:120) to go to 50 and from (1500:1520) to go to -100 , we chose this interval to show clearly the noise in the system.

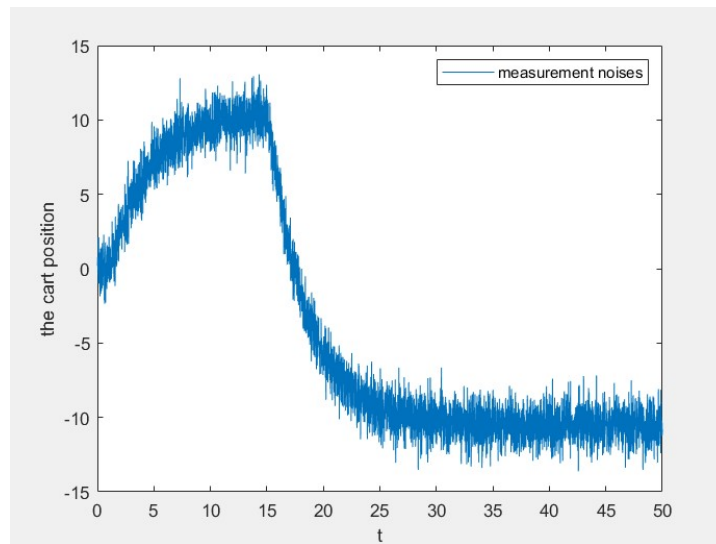


Figure 3.5: Simulation of the estimated system

The fig 3.5 shows the measurement y as you see it is a fussy measurement because of the disturbances and the noises, to compare it and to understand the noises and disturbances we need to plot the real measurement with disturbances but without noises (see 3.6).

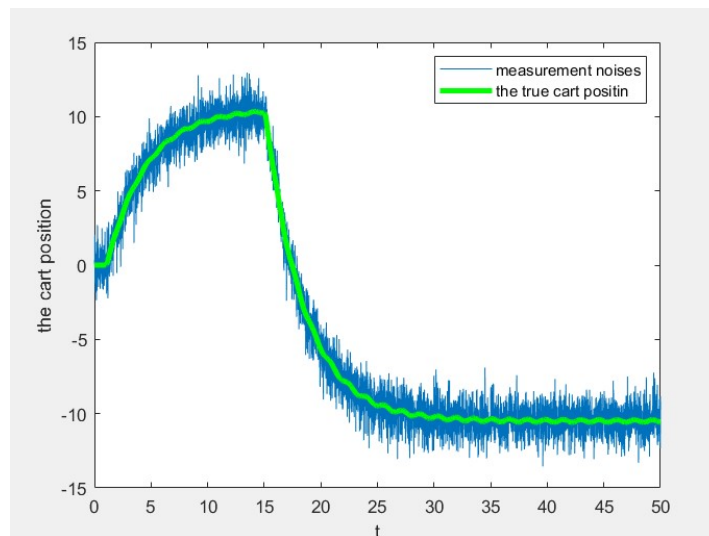


Figure 3.6: Simulation of the estimated system and the real

To see the \hat{x} we need to enter the input u and y to the kalman filter, and after doing that we get this simulation

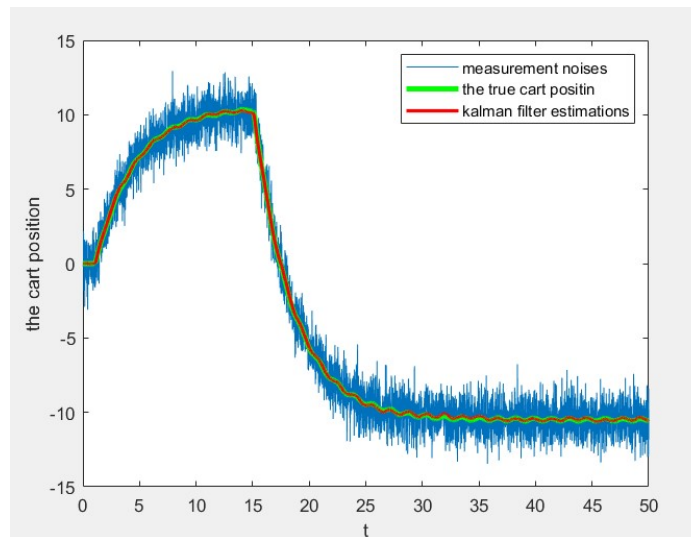


Figure 3.7: Simulation of the estimated system and the real with kalman filter

Where the red one is the kalman filter, and you see in fig 3.5 the kalman filter almost lies on top of the real measurement (see fig 3.8 to see the difference between the real measurement and the kalman filter estimation)

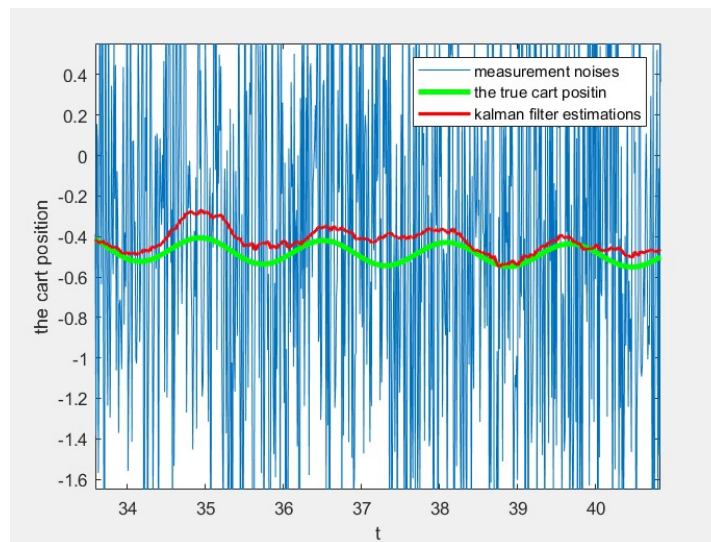


Figure 3.8: The difference between the real measurement and the kalman filter estimation

As you see in the fig 3.8 the kalman filter estimate the system with noise and the result are almost similar to the real measurement

3.6 Conclusion

In this chapter we simulate our system and the control strategies via Matlab, we explained how to use and write codes, and we interpreted the results. By reaching the simulation we reach our main goal to see and study the difference between automation control.

General conclusion

An inverted pendulum on a cart is a typical nonlinear system with an unstable equilibrium point. This has been used a test bed to evaluate the performance of a controller, there are several approaches for the control of a nonlinear system. But first the determination of the dynamics of the system is obligated and once a system is represented in a linear form, we can apply any conventional linear control method as we did in the first chapter, after that we described in theoretical way about swing-up control using non-linear controller, the stabilisation control of one inverted pendulum, and then we developed it to get the controller for two inverted pendulum. We went through control laws theoretically we talked about the pole placement method and the linear quadratic regulator we talked also about the kalman filter for the full state estimation to estimate the noises and disturbances and to make the cart go to the desired point smoothly. In the last, we tried to simulate different control strategies of one inverted pendulum system that we talked about we saw the difficulties, requirement of each control strategies and even the results of each one and the effect of them on the cart position and pendulum and their speeds.

Our system is among the classic mechanical systems that have been studied extensively in control theory during the last four decades. This system was originally used as a benchmark for educational purposes.

This work makes me passionate about control strategies. I think that I can do better in this domain and develop this work to another level. In the future I will try the other control strategies and control law that I studied in these years. I am passionate about the AI and neural networking,

I will try to apply it in this work at least in the simulation. I will develop my pendulum into two inverted pendulums.

Appendix

0 ODE45 MAtlab

```
function dx = invertedpendcart(t, x)

M=3;
m=0.4;
l=0.7;
g=-9.8;
dc = 1;
dp = 0.1;
U=0;
Cx = cos(x(3));
Sx = sin(x(3));

dx = zeros(4,1);
dx(1)=x(2);
dx(2)=((- (m^2)*(l^2)*g*Sx*Cx)+((m^2)*(l^3)*Sx*(x(4)^2))- (m*(l^2)*dc*x(2))
+(m*l*Cx*dp*x(4))+(m*(l^2)*U))/((m*(l^2))*(M+(m*(Sx^2))));
```

```
dx(3)=x(4);
dx(4)=((- (m^2)*(l^2)*Sx*Cx*(x(4)^2))+(m*l*dc*Cx*x(2))-((M+m)*dp*x(4))+((M+m)*g*m*l*Sx)
-(m*l*Cx*U))/((m*(l^2))*(M+(m*(Sx^2))));

%%

clc

clear all

close all

%%

f1=@invertedpendcart;

%%

Tf=10;

T=[0:0.01:Tf];

%%

x0=[0.5 0 5*pi/6 0]

%%

% K=[-35 -34 -150 -34];

%%

[t,x]=ode45(f1, T, x0);

%% PLOT

figure(1)

subplot(2,2,1)

plot(t, x(:,1))

xlabel('t')

ylabel('x(t)')

title('x')

grid
```

```
subplot(2,2,2)
plot(t, x(:,2))
xlabel('t')
ylabel('dot(x)(t)')
title('dot(x)')
grid
```

```
subplot(2,2,3)
plot(t, x(:,3))
xlabel('t')
ylabel('\theta(t)')
title('\theta')
grid
```

```
subplot(2,2,4)
plot(t, x(:,4))
title('theta dot')
xlabel('t')
ylabel('theta dot')
```

```
grid
```

```
% %% animation
%
% X0=0;
% Y0=0;
%
% for i=1:length(t)
```

```

%   Xc=x(i,1);
%   Xp=x(i,1)+l*(sin(pi-x(i,3)));
%   Yp=l*cos(pi-x(i,3));
%
%
%   figure(2)
%   plot([-3 5],[0 0],'linewidth',6,'color','g');
%   axis([-4 6 -4 4]);
%   line([Xc Xp],[Y0 Yp],'linewidth',2,'color','b');
%   hold on
%   plot(Xc,Y0,'s','markersize',30,'markerfacecolor','y');
%   plot(Xp,Y0,'o','markersize',15,'markerfacecolor','r');
%   hold off
% end

```

0 Pole Placement

```
clear all, close all, clc
```

```
M=3;
```

```
m=0.4;
```

```
l=0.7;
```

```
g=-9.8;
```

```
dc = 0.1;
```

```
dp = 0.1;
```

```
s = 1;
```

```
A= [ 0 1 0 0;
```

```
    0 -dc/M (-m*g)/M 0;
```

```

0 0 0 1;
0 -(s*dc)/(M*1) -(s*(m+M)*g)/(M*1) 0];

```

```
B= [0; 1/M; 0; s*1/(M*1)];
```

```

eig(A)
plot(eig(A),'r*','LineWidth',2)
grid on
ctrb(A,B);

```

```

rank(ctrb(A,B));
eigs = [-3; -1.5; -2; -2.5];

```

```
k = place(A,B,eigs);
```

```
eig(A-B*k);
```

0 Pole Placement drawn

```
function invertpend(t , x, k)
```

```
function v=f(x,u)
```

```
    v=[x(2);(m*sin(x(3))*(g*cos(x(3))-1*x(4)^2)+u)/(M+m*sin(x(3))^2);
```

```
        x(4);
```

```
        (sin(x(3))*((M+m)*g-m*1*x(3)^2*cos(x(3)))+u*cos(x(3)))/(1*(M+m*sin(x(3))^2))];
```

```
end;
```

```
function draw(x)
```

```

    clf(); hold on; axis('off'); axis([-3,3,-3,3]);
    plot([ x(1), x(1)-1*sin(x(3))], [0, 1*cos(x(3))], 'blue', 'LineWidth', 2);
    plot(x(1)+[-.5,.5,.5,-.5,-.5], [0,0,-.25,-.25,0], 'black', 'LineWidth', 3);
    drawnow()
end
l=1; dt=0.01; M=5; g=9.81 ; m=1;
A=[0 1 0 0;0 0 (m*g)/M 0;0 0 0 1;0 0 (M+m)*g/(1*M) 0];
B=[0 ; 1/M ; 0 ; 1/(1*M)];
K=place(A,B,[-2 -2.1 -2.2 -2.3])
x=[0;0;0.5;0];
for t=0:dt:5
    u=-K*x;
    x=x+dt*f(x,u);
    draw(x);
end
end

```

0 LQR code

```
clear all, close all, clc
```

```

M=3;
m=0.4;
l=0.7;
g=9.8;
dc = 1;
dp = 0.1;
s = 1;

```

```
A= [ 0 1 0 0;  
    0 -dc/M (-m*g)/M 0;  
    0 0 0 1;  
    0 -(s*dc)/(M*1) -(s*(m+M)*g)/(M*1) 0];
```

```
B= [0; 1/M; 0; s*1/(M*1)];
```

```
ctrb(A,B)
```

```
rank(ctrb(A,B))
```

```
Q= [1 0 0 0;  
    0 1 0 0;  
    0 0 10 0;  
    0 0 0 10];
```

```
R= 0.001;
```

```
k = lqr(A,B,Q,R)
```

```
eig(A-B*k);
```

```
[T,D] = eig(A-B*k);
```

```
diag(real(D));
```

```
T(:,1);
```

```
% eigs = [-1.1; -1.2; -1.4; -1.3]
```

```
% k = place(A,B,eigs)
```


0 Kalman filter

```
clear all, close all, clc

m = 0.4 ;
M = 3 ;
l = 0.7 ;
g = -9.8 ;
dc= 1 ;
s = -1;

A= [ 0 1 0 0;
     0 -dc/M (-m*g)/M 0;
     0 0 0 1;
     0 -(s*dc)/(M*l) -(s*(m+M)*g)/(M*l) 0];

B= [0; 1/M; 0; s*1/(M*l)];
C= [1 0 0 0];
D= zeros(size(C,1),size(B,2));
rank(observ(A,C));

%% augment system + disturbance + noise
Vd= 0.1*eye(4);
Vn = 1;

BF = [B Vd 0*B];
sysC = ss(A,BF,C,[0 0 0 0 0 Vn]);
```

```
sysFullOutput = ss(A,BF,eye(4),zeros(4,size(BF,2)));
```

```
%% building a kalman filter
```

```
% [Kf,P,E] = lqe(A,Vd,C,Vd,Vn);
```

```
Kf= (lqr(A',C',Vd,Vn))';
```

```
sysKF = ss(A-Kf*C,[B Kf],eye(4),0*[B Kf]);
```

```
%% Estimation of the system
```

```
dt = 0.01;
```

```
t = dt:dt:50;
```

```
uDIST = randn(4,size(t,2));
```

```
uNOISE = randn(size(t));
```

```
u = 0*t;
```

```
u(100:120) = 50;
```

```
u(1500:1520) = -100;
```

```
uAUG = [u; Vd*Vd*uDIST; uNOISE];
```

```
[y,t] = lsim(sysC,uAUG,t);
```

```
plot(t,y);
```

```
    xlabel('t')
```

```
    ylabel('the cart position')
```

```
%% the true cart position without noise
```

```
[xtrue,t] = lsim(sysFullOutput,uAUG,t);  
hold on  
    plot(t,xtrue(:,1),'g','LineWidth',3)  
  
%% kalman filter  
[x,t] = lsim(sysKF,[u; y'],t);  
    plot(t,x(:,1),'RED ','LineWidth',2);  
    legend('measurement noises','the true cart positin','kalman filter estimations')  
plot(t,xtrue,'blue',t,x,'green','LineWidth')
```

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